Integrating Proxy Theories and Numeric Model Lifting for Floating-Point Arithmetic

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Why Floating-Point Arithmetic?

Floating-point (FP) = practical approximation of real numbers

- Finite representation on computers
- Dynamic range
- Speed, implementation in hardware
FP arithmetic different from Real arithmetic

IEEE 754 (2008) Standard says:

\[ x \, op_F \, y = \text{round}(x \, op_R \, y) \]

Standard describes 5 rounding modes
IEEE 754 (2008) Standard says:

\[ x \ op_F y = \text{round}(x \ op_R y) \]

Standard describes 5 rounding modes

Examples of formulas satisfiable in FP:

- \[ x \oplus y = x \land y > 0 \]
- \[ x \ominus (y \ominus z) > (x \ominus y) \ominus z \]
- \[ x \otimes (y \ominus z) > (x \otimes y) \ominus x \otimes z \]
Floating-point reasoning: approaches

- Traditionally: theorem proving, abstract domains
- More recently: decision procedures
  - Examples: Mathsat, z3
  - Big win: witness generation
  - Technique: bit-blasting, bit-vectors
  - Limitation: leads to huge boolean encodings
Automatic Detection of Floating-Point Exceptions

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Using Reduced Precision FP [IJCAR14]

Solving FP formula \( f \)

- \( f' = \text{reduce\_precision}(f) \)
- while \( f' \neq f \)
  - if \( \exists \sigma : \sigma \models f' \)
    - if \( \sigma \models f \)
      - return \( \sigma \)
    - else
      - increase precision of \( f' \)
Consider $f$:
Solve instead:

$$(x \oplus y) \oplus z > x \oplus (y \oplus z)$$
$$(x \ominus_9 y) \oplus_9 z > x \ominus_9 (y \ominus_9 z)$$

Satisfiable in $\text{FP}_9$ (as any bit-blaster will tell you):

$x_0 = y_0 = 1.18, \quad z_0 = 1.97 \times 10^{-3}$

Problem: $f(x_0, y_0, z_0) \rightarrow \text{false}! \quad \text{What now?}$
Proxy solution

- Proxy solution gets discarded [IJCAR14] if it does not work as is:
  - effort wasted

Can we use the proxy solution in some way?

Can the proxy solution be lifted to an actual satisfying solution?
Lifting a proxy solution

Solving FP formula $f$

- $f' = \text{reduce\_precision}(f)$
- while($f' \neq f$)
  - if $\exists \sigma : \sigma \models f'$
    - if $\sigma \models f$
      - return $\sigma$
    - else
      - do\_something($\sigma$)
  - else
    - increase precision of $f'$
Framework: Overview

$f_T := f$ mapped to $T$

$\exists \sigma_T. \sigma_T \models_T f_T$?

$\sigma := \text{ToFloat}(\sigma_T)$

$\sigma \models f$?

$\sigma := \text{ModelLift}(\sigma, \sigma_T, f_T)$

$\exists \sigma_T. \sigma_T \models_T f_T$? (success)

$f_T := \text{Refine}(f_T)$ (failure)

$\sigma$ (exception)

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Proxy Theories and Model Lifting for Floating-Point Arithmetic
Proxy theories for floating-point: Conditions

- offer a mapping from FP formulas
- easier to reason about than FP
- offer a mapping to FP models
- gradually refinable back to FP
Proxy theories for floating-point: Candidates

Reduced precision (reduced exponent + mantissa) FP
- “easier”
- map solutions to original precision FP by padding bits
- refine by gradually increasing exponent, mantissa

Real arithmetic
- sometimes easier
- map solutions to FP by rounding
- refine by interpreting some real operators as FP [DATE14]
Numeric Model Lifting
Framework: overview

- $f_T := f$ mapped to $T$
- $\exists \sigma_T. \sigma_T \models_T f_T$?
- $\sigma := ToFloat(\sigma_T)$
- $\sigma \models f$?
- $\sigma := ModelLift(\sigma, \sigma_T, f_T)$
- $f_T := \text{Refine}(f_T)$

Flowchart:
- Yes: $\sigma$, continue
- No: $f_T := \text{Refine}(f_T)$, success or failure
- Exception: UNSAT

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Proxy Theories and Model Lifting for Floating-Point Arithmetic
**Assumption:** proxy theory $T$ delivers satisfying $T$ assignment such that an FP solution is nearby

**Idea for lifting proxy soln to FP soln:**
- $T$ assign. gives satisfying Boolean skeleton
- fix constraints where $T$ and FP disagree
- pick small subset $\text{Vars}(f)$ to do so, keep others constant
**Numeric Model Lifting: Example**

\[ f(x, y) \equiv x \otimes y \otimes y \oplus 2 \lor x \oplus y \ominus 0 \]

- **\( \mathbb{R} : x \ast y^2 > 2 \lor x + y < 0 : \bar{x} = 1, \bar{y} = 1.42 \)**
- **\( \ln \mathbb{R} : T \lor F = T \)  \( \ln \text{FP} : F \lor F = F \)**
- **Goal:** fix FP assign. so that \( x \otimes y \otimes y \oplus 2 = T \)

\[ f'(x) \equiv x \otimes \bar{y} \otimes \bar{y} \oplus 2 \land \neg(x \oplus \bar{y} \ominus 0) \]

\( f' \) is: (i) univariate, (ii) linear, (iii) conjunctive
1. Reduces decision problem \((f)\) to simpler one \((f')\)
2. Uses off-the-shelf floating-point SMT solver for \(f'\)

**Benefits:**

- Propositional structure of \(f\) reduced to conjunction
- Typically, \(\text{Vars}(f') \subseteq \text{Vars}(f)\)
- Often, degree \(\text{deg}(f') < \text{deg}(f)\)
- Independent of where proxy solution came from
Framework: On Soundness, Termination, Completeness

\[ f_T := f \text{ mapped to } T \]

\[ \exists \sigma_T. \sigma_T \models_T f_T? \]

\[ f_T := \text{Refine}(f_T) \]

\[ \sigma := \text{ToFloat}(\sigma_T) \]

\[ \sigma \models f \]

\[ \sigma := \text{ModelLift}(\sigma, \sigma_T, f_T) \]

Failure

Success

Exception

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Proxy Theories and Model Lifting for Floating-Point Arithmetic
Experimental Evaluation
Set I:
- Non-linear benchmarks [FMSD14]
- Ignored casts (single precision), ignored special values
- Benchmarks are satisfiable or status is unknown

Set II:
- False identity non-linear benchmarks, $E - \hat{E} > \epsilon$
  e.g., $(a^2 \ominus b^2) - (a \ominus b)(a \oplus b) > \epsilon$
- is of interest in compiler optimization
- single precision

Timeout: 20 min
# Experimental Evaluation (Set II)

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<th>Time (s)</th>
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### Experimental Evaluation (Results)

**Set I: total 22, Set II: total 15**

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<th>Mathsat</th>
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Future Directions

- Non-symbolic model lifting
- Numeric solvers for approximate solutions
- Handling other combinations of proxy ↔ actual solutions
  - UNSAT ↔ UNSAT
  - UNSAT ↔ SAT
  - SAT ↔ UNSAT
Thank You!
Backup Slides
### Experimental Evaluation (Set I)

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Instantiation with Real Arithmetic Proxy Theory

Set III:

- $E > \hat{E}$
  $$(((a_1 \oplus a_2) \oplus (a_3 \oplus a_4)) \oplus a_5) > (((a_1 \oplus a_2) \oplus a_3) \oplus a_4) \oplus a_5$$

- $(0, 1024.0]$  
- single precision, RoundToNearestEven
- Offset $O$ is singleton (gradient analysis)
## Experimental Evaluation

### Set III benchmarks

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