Introduction

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Motivation satisfiability solving

From 100 variables, 200 clauses (early 90's)
to 1,000,000 vars. and 5,000,000 clauses in 15 years.

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SAT used to solve many other problems!

Overview

- Introduction
- The Satisfiability problem
- Terminology
- SAT solving
- SAT benchmarks

"You are chief of protocol for the embassy ball. The crown prince instructs you either to invite *Peru* or to exclude *Qatar*. The queen asks you to invite either *Qatar* or *Romania* or both. The king, in a spiteful mood, wants to snub either *Romania* or *Peru* or both. Is there a guest list that will satisfy the whims of the entire royal family?"

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$$(P \vee \neg Q) \wedge (Q \vee R) \wedge (\neg R \vee \neg P)$$

Truth Table

$$F := (P \lor \neg Q) \land (Q \lor R) \land (\neg R \lor \neg P)$$

P	Q	R	falsifies	$\varphi \circ F$
0	0	0	$(Q \lor R)$	0
0	0	1	—	1
0	1	0	$(P \lor \neg Q)$	0
0	1	1	$(P \lor \neg Q)$	0
1	0	0	$(Q \lor R)$	0
1	0	1	$(\neg R \lor \neg P)$	0
1	1	0	—	1
1	1	1	$(\neg R \lor \neg P)$	0

Slightly Harder Example

What are the solutions for the following formula?

$$(A \lor B \lor \neg C)$$
$$(\neg A \lor \neg B \lor C)$$
$$(B \lor C \lor \neg D)$$
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A	B	C	D	A	B	C	D
0	0	0	0	1	0	0	0
0	0	0	1	1	0	0	1
0	0	1	0	1	0	1	0
0	0	1	1	1	0	1	1
0	1	0	0	1	1	0	0
0	1	0	1	1	1	0	1
0	1	1	0	1	1	1	0
0	1	1	1	1	1	1	1

Given a *CNF formula*,
does there exist an *assignment*
to the *Boolean variables*
that satisfies all *clauses*?

Boolean variable

- can be assigned the Boolean values 0 or 1

Literal

- refers either to x_i or its complement $\neg x_i$
- literals x_i are satisfied if variable x_i is assigned to 1 (true)
- literals $\neg x_i$ are satisfied if variable x_i is assigned to 0 (false)

Terminology: Clauses

Clause

- Disjunction of literals: E.g. $C_j = (l_1 \lor l_2 \lor l_3)$
- Can be falsified with only *one* assignment to its literals: All literals assigned to false
- Can be satisfied with $2^k - 1$ assignment to its k literals
- One special clause - the empty clause (denoted by $\emptyset$) - which is always falsified

Terminology: Formulae

Formula

- Conjunction of clauses: E.g. $\mathcal{F} = C_1 \land C_2 \land C_3$
- Is *satisfiable* if there exists an assignment satisfying all clauses, otherwise *unsatisfiable*
- Formulae are defined in *Conjunction Normal Form* (CNF) and generally also stored as such - also learned information

Assignment

- Mapping of the values 0 and 1 to the variables
- $\varphi \circ \mathcal{F}$ results in a reduced formula $\mathcal{F}_{\text{reduced}}$:
  - all satisfied clauses are removed
  - all falsified literals are removed
- *satisfying assignment* $\leftrightarrow$ $\mathcal{F}_{\text{reduced}}$ is empty
- *falsifying assignment* $\leftrightarrow$ $\mathcal{F}_{\text{reduced}}$ contains $\emptyset$
- *partial assignment* versus *full assignment*

Resolution

Given two clauses $C_1 = (x \vee a_1 \vee \cdots \vee a_n)$ and $C_2 = (\neg x \vee b_1 \vee \cdots \vee b_m)$, the resolvent of C_1 and C_2 (denoted by $C_1 \bowtie C_2$) is $R = (a_1 \vee \cdots \vee a_n \vee b_1 \vee \cdots \vee b_m)$

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Examples for $F := (P \vee \neg Q) \wedge (Q \vee R) \wedge (\neg R \vee \neg P)$

- $(P \vee \neg Q) \bowtie (Q \vee R) = (P \vee R)$
- $(P \vee \neg Q) \bowtie (\neg R \vee \neg P) = (\neg Q \vee \neg R)$
- $(Q \vee R) \bowtie (\neg R \vee \neg P) = (Q \vee \neg P)$

Resolution

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Resolution, i.e., adding resolvents until fixpoint, is a complete proof procedure. It produces the empty clause if and only if the formula is unsatisfiable

A clause C is a tautology if it contains
for some variable x, both the literals x and $\neg x$.

Slightly Harder Example 2

Compute all non-tautological resolvents for:

$$(A \lor B \lor \neg C) \land (\neg A \lor \neg B \lor C) \land \\ (B \lor C \lor \neg D) \land (\neg B \lor \neg C \lor D) \land \\ (A \lor C \lor D) \land (\neg A \lor \neg C \lor \neg D) \land \\ (\neg A \lor B \lor D)$$

Which resolvents remain after removing the supersets?

SAT solving: Unit propagation

A *unit clause* is a clause of size 1

UnitPropagation $(\varphi, \mathcal{F})$:

while $\emptyset \notin \mathcal{F}$ **and** unit clause y exists **do**
    expand φ and simplify $\mathcal{F}$
end while
return $\varphi, \mathcal{F}$

Unit propagation: Example

$$\mathcal{F}_{\text{unit}} := (\neg x_1 \lor \neg x_3 \lor x_4) \land (\neg x_1 \lor \neg x_2 \lor x_3) \land (\neg x_1 \lor x_2) \land (x_1 \lor x_3 \lor x_6) \land (\neg x_1 \lor x_4 \lor \neg x_5) \land (x_1 \lor \neg x_6) \land (x_4 \lor x_5 \lor x_6) \land (x_5 \lor \neg x_6)$$

Unit propagation: Example

$$\mathcal{F}_{\text{unit}} := (\neg x_1 \vee \neg x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2) \wedge (x_1 \vee x_3 \vee x_6) \wedge (\neg x_1 \vee x_4 \vee \neg x_5) \wedge (x_1 \vee \neg x_6) \wedge (x_4 \vee x_5 \vee x_6) \wedge (x_5 \vee \neg x_6)$$

$$\varphi = \{x_1{=}1\}$$

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$$\varphi = \{x_1{=}1, x_2{=}1\}$$

Unit propagation: Example

$$\mathcal{F}_{\text{unit}} := (\neg x_1 \vee \neg x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2) \wedge (x_1 \vee x_3 \vee x_6) \wedge (\neg x_1 \vee x_4 \vee \neg x_5) \wedge (x_1 \vee \neg x_6) \wedge (x_4 \vee x_5 \vee x_6) \wedge (x_5 \vee \neg x_6)$$

$$\varphi = \{x_1{=}1, x_2{=}1, x_3{=}1\}$$

Unit propagation: Example

$$\mathcal{F}_{\text{unit}} := (\neg x_1 \lor \neg x_3 \lor x_4) \land (\neg x_1 \lor \neg x_2 \lor x_3) \land (\neg x_1 \lor x_2) \land (x_1 \lor x_3 \lor x_6) \land (\neg x_1 \lor x_4 \lor \neg x_5) \land (x_1 \lor \neg x_6) \land (x_4 \lor x_5 \lor x_6) \land (x_5 \lor \neg x_6)$$

$$\varphi = \{x_1{=}1, x_2{=}1, x_3{=}1, x_4{=}1\}$$

Davis Putnam Logemann Loveland [DP60,DLL62]

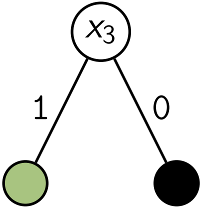
- Simplify (Unit Propagation)
- Split the formula
    - Variable Selection Heuristics
    - Direction heuristics

DPLL: Example

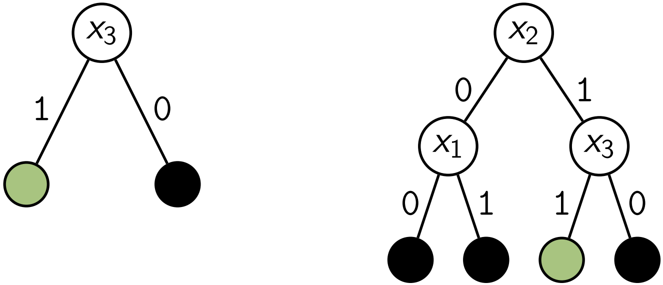
$$\mathcal{F}_{\text{DPLL}} := (x_1 \lor x_2 \lor \neg x_3) \land (\neg x_1 \lor x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2 \lor x_3) \land (x_1 \lor x_3) \land (\neg x_1 \lor \neg x_3)$$

DPLL: Example

$$\mathcal{F}_{\text{DPLL}} := (x_1 \lor x_2 \lor \neg x_3) \land (\neg x_1 \lor x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2 \lor x_3) \land (x_1 \lor x_3) \land (\neg x_1 \lor \neg x_3)$$



$$\mathcal{F}_{\text{DPLL}} := (x_1 \lor x_2 \lor \neg x_3) \land (\neg x_1 \lor x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2 \lor x_3) \land (x_1 \lor x_3) \land (\neg x_1 \lor \neg x_3)$$



Slightly Harder Example 3

Construct a DPLL tree for:

$$(A \vee B \vee \neg C) \wedge (\neg A \vee \neg B \vee C) \wedge$$
$$(B \vee C \vee \neg D) \wedge (\neg B \vee \neg C \vee D) \wedge$$
$$(A \vee C \vee D) \wedge (\neg A \vee \neg C \vee \neg D) \wedge$$
$$(\neg A \vee B \vee D)$$

SAT solving: Decisions, implications

Decision variables

- Selected by the heuristics
- Play a crucial role in performance

Implied variables

- Assigned by reasoning (e.g. unit propagation)
- Maximizing the number of implied variables is an important aspect of look-ahead SAT solvers

SAT Solving: Clauses $\leftrightarrow$ assignments

- A clause C represents a set of falsified assignments, i.e. those assignments that falsify all literals in C
- A falsifying assignment φ for a given formula represents a set of clauses that follow from the formula
  - For instance with all decision variables
  - Important feature of conflict-driven SAT solvers

SAT solvers

Conflict-driven

- "brute-force", complete
- examples: zchaff, minisat, rsat

Look-ahead

- lots of reasoning, complete
- examples: march, OKsolver, kcnfs

Local search

- local optimazations, incomplete
- examples: WalkSAT, UnitWalk

Applications: Industrial

- Model Checking
  - Turing award '07 Clarke, Emerson, and Sifakis
- Software Verification
- Hardware Verification
- Equivalence Checking Problems

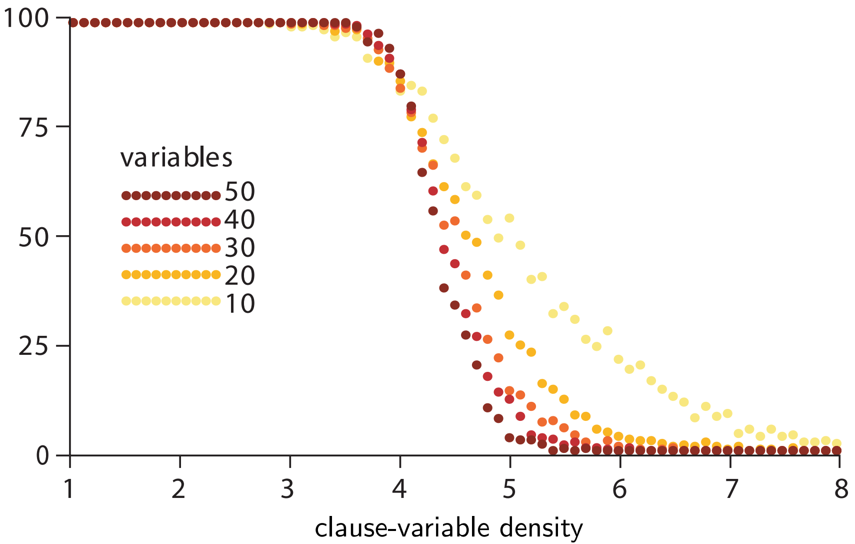
Applications: Crafted

- Combinatorial problems
- Sudoku
- Factorization problems

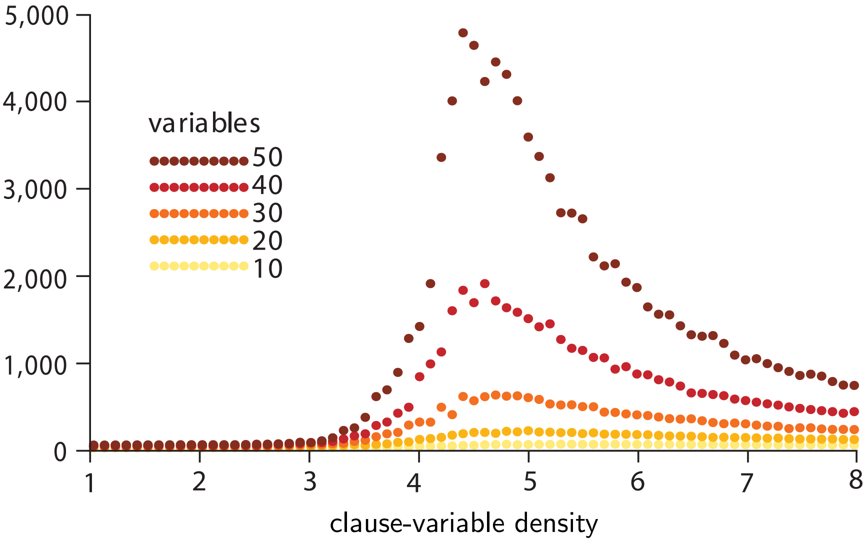
Random k-SAT: Introduction

- All clauses have length k
- Variables have the same probability to occur
- Each literal is negated with probability of 50%
- Density is ratio Clauses to Variables

Random 3-SAT: % satisfiable, the phase transition



Random 3-SAT: exponential runtime, the threshold



SAT game

by Olivier Roussel

http://www.cs.utexas.edu/~marijn/game/

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