Fast and Accurate Low Rank Approximation of Massive Graphs

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Outline

- **Some Problems in Network Analysis**
  - Link prediction problem
  - Link prediction with multiple sources of information
  - Affiliation recommendation problem
- **Tools and Methods**
  - Path based methods for link prediction
  - Low rank approximations
  - Graph clustering
  - Clustered low rank approximation
- **Experimental Results**
- **Summary and Conclusions**

Networks in Applications

- Web graphs
- Collaboration and citation networks
- Social networks (Facebook, MySpace, LiveJournal, ...)
- Call graph networks (SKT, AT&T)
- Networks in Bioinformatics
- ...

Graphs

- Graph $G = (\mathcal{V}, \mathcal{E})$,
  $\mathcal{V}$ is a set of vertices, and
  $\mathcal{E}$ is a set of edges between vertices.
- The adjacency matrix $A = [a_{ij}]$ of G is given by

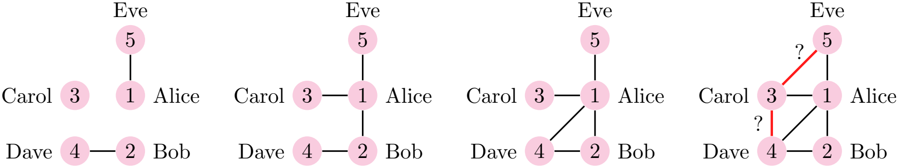
$$\begin{aligned} a_{ij} &= w_{ij} && \text{if there is an edge between vertex } i \text{ and } j \\ a_{ij} &= 0 && \text{otherwise} \end{aligned}$$

- Undirected graph then A is symmetric
- Directed graph then A is non-symmetric

Link Prediction Problem

Consider a social network that evolves with time,

$$\cdots \longrightarrow A^{(t-2)} \longrightarrow A^{(t-1)} \longrightarrow A^{(t)} \xrightarrow{?} A^{t+1}$$

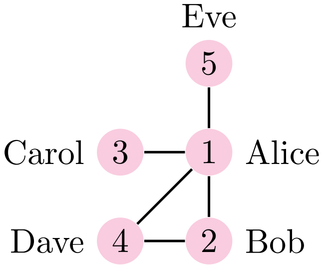


Q: Can we predict links that will form at time step $t+1$?

A: Many models exist, e.g. common neighbors, the Katz measure, etc.

Link prediction with multiple sources of information

In most cases, we have a social network between users

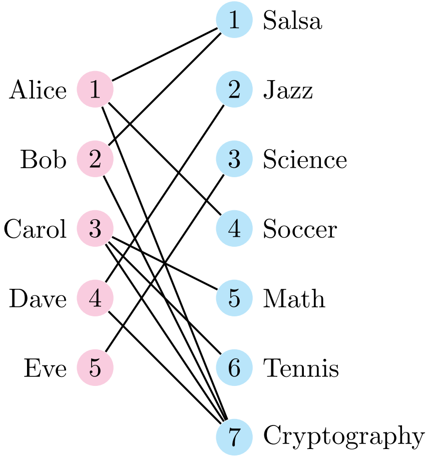


But also additional sources of information

- Other types of links between users, e.g. family ties, affiliations
- User profiles, demographic information
- Blog postings by users, movie ratings, etc
- ...

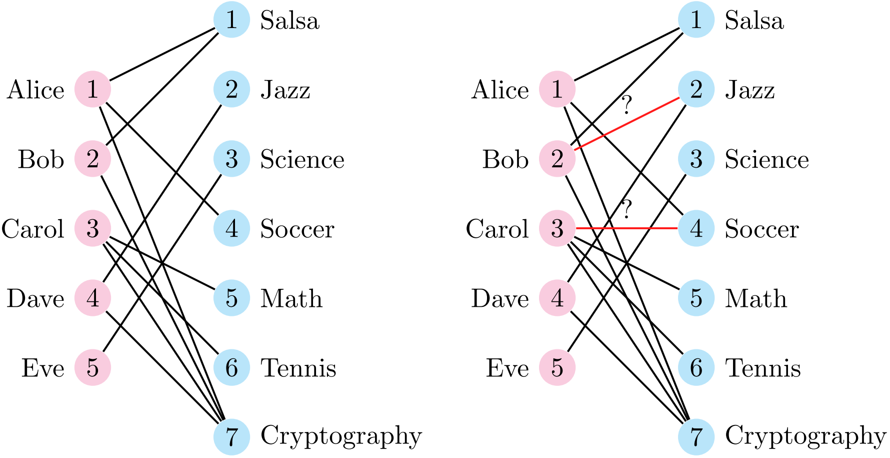
Affiliation recommendation/prediction

Consider a users $\times$ affiliations network



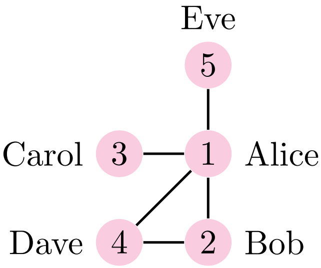
Affiliation recommendation

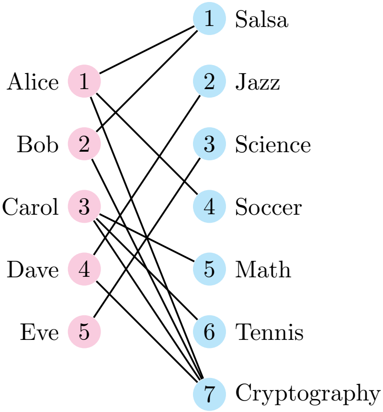
We want to suggest groups/affiliations to users



Affiliation recommendation given a social network

Suppose we additionally have a social network:





Can the social network help to improve affiliation recommendations?

Challenges

- Huge number of users and/or affiliations, e.g. Facebook, MySpace, Orkut, LiveJournal
- Multiple sources of information
- Need scalable algorithms and “accurate” predictions

Tools and Methods

Path based methods for link prediction

- No. of common neighbors can be computed as:

$$P_{\text{cn}} = A^2$$

  Yields no. of paths of length 2 between any two vertices

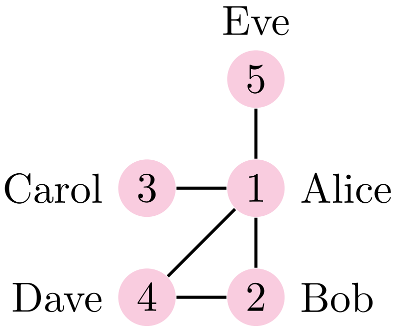
- Katz measure and truncated Katz measure

$$P_{\text{katz}} = \beta A + \beta^2 A^2 + \beta^3 A^3 + \cdots = \sum_{i=1}^{\infty} \beta^i A^i = (I - \beta A)^{-1} - I$$

$$P_{\text{t-katz}} = \beta A + \beta^2 A^2 + \cdots + \beta^k A^k = \sum_{i=1}^{k} \beta^i A^i$$

  - A^3 gives no. of paths of length 3, A^4 is no. of paths of length 4, & so on
  - Small β penalizes paths with larger lengths
  - Large β amplifies paths with larger lengths

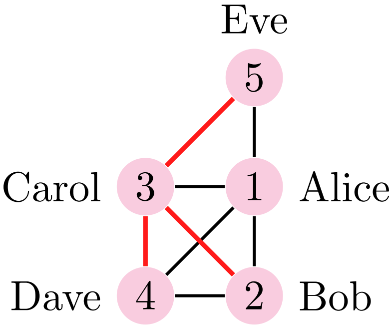
A social network and its adjacency matrix



$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Link prediction based on common neighbors

- No. of common neighbors is given by A^2
- Carol: common neighbor predicts links to Bob, Dave and Eve (in red).

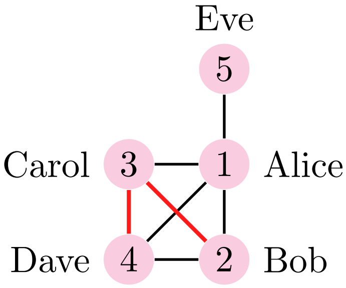


$$A^2 = \begin{bmatrix} 4 & 1 & 0 & 1 & 0 \\ 1 & 2 & \color{red}{1} & 1 & \color{red}{1} \\ 0 & \color{red}{1} & 1 & \color{red}{1} & \color{red}{1} \\ 1 & 1 & \color{red}{1} & 2 & \color{red}{1} \\ 0 & \color{red}{1} & \color{red}{1} & \color{red}{1} & 1 \end{bmatrix}$$

Link prediction based on Katz measure

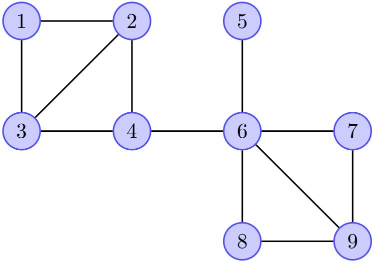
$$K(A,\beta) = \beta A + \beta^2 A^2 + \beta^3 A^3 + \cdots = (I - \beta A)^{-1} - I$$

- With $\beta = 0.4$
- Top two predictions for Carol are to Bob and Dave (in red)
- Katz score between Carol & Eve is smaller than other scores (in green)



$$K(A, 0.4) = \begin{bmatrix} * & * & * & * & * \\ * & * & 1.8 & * & 1.8 \\ * & 1.8 & * & 1.8 & 1.1 \\ * & * & 1.8 & * & 1.8 \\ * & 1.8 & 1.1 & 1.8 & * \end{bmatrix}$$

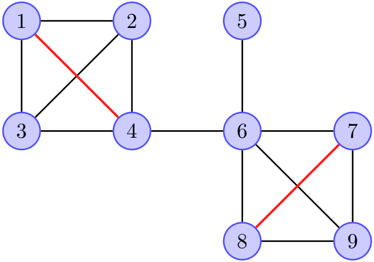
After a few time steps....



$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Common Neighbors

- All possible predictions in red, based on common neighbors.
- Possible recommendation: top two highest scores

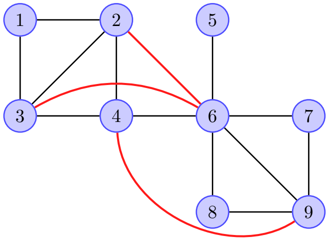


$$A^2 = \begin{bmatrix} 2 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 3 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 5 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 & 1 & 1 & 3 \end{bmatrix}$$

Katz measure

$$Katz(A,\beta) = \sum_{i=1}^{\infty} \beta^i A^i = (I - \beta A)^{-1} - I, \qquad \beta < |\lambda_i| \quad \forall\ i$$

- With $\beta = 1/3$ top three Katz scores are the $\color{red}{X}$'s



$$\begin{bmatrix}
* & * & * & * & * & * & * & * & * \\
* & * & * & * & * & \color{red}{X} & * & * & * \\
* & * & * & * & * & \color{red}{X} & * & * & * \\
* & * & * & * & * & * & * & * & \color{red}{X} \\
* & * & * & * & * & * & * & * & * \\
* & \color{red}{X} & \color{red}{X} & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * \\
* & * & * & \color{red}{X} & * & * & * & * & *
\end{bmatrix}$$

Importance of low rank approximations

- Enables scalability
- Can filter out noise
- Introduces interpretable features...
- For example, with the spectral approximation $A \approx V \Lambda V^{\mathsf{T}}$,

$$A^k \approx (VDV^{\mathsf{T}})^k = (VDV^{\mathsf{T}})(VDV^{\mathsf{T}}) \cdots (VDV^{\mathsf{T}}) = VD^kV^{\mathsf{T}}$$

Link predictors with low rank approximations

- With spectral approximation,

$$A \approx V \Lambda V^{\mathsf{T}}$$

- Common neighbor,

$$P_{\text{cn}} = A^2, \qquad P_{\text{cn-lr}} = V \Lambda^2 V^{\mathsf{T}}$$

- Katz measure,

$$P_{\text{katz}} = (I - \beta A)^{-1} - I = \sum_{i=1}^{\infty} \beta^i A^i \qquad P_{\text{t-katz}} = \sum_{i=1}^{k} \beta^i A^i$$

$$P_{\text{katz-lr}} = V \left((I - \beta \Lambda)^{-1} - I \right) V^{\mathsf{T}} \qquad P_{\text{t-katz-lr}} = \sum_{i=1}^{k} \beta^i V \Lambda^i V^{\mathsf{T}}$$

Generalized Katz, evolving graphs and model learning

Replace β in truncated Katz with $\alpha_1, \ldots, \alpha_k$

$$P_{\text{t-katz}} = \sum_{i=1}^{k} \beta^i A^i \quad \Rightarrow \quad P_{\text{gt-katz}} = \sum_{i=1}^{k} \alpha_i A^i$$

Social network graphs are evolving....

- A adjacency matrix at time step t_1
- B adjacency matrix at time step $t_2 > t_1$
- Usually $B = A + \Delta$

Learn α_i from snapshots A and B

$$\min_{\alpha_i} \left\| \sum_{i=1}^{k} \alpha_i A^i - B \right\|_F \quad \text{or rather} \quad \min_{\alpha_i} \left\| \left(\sum_{i=1}^{k} \alpha_i A^i - B \right) \cdot M \right\|_F$$

where $M_{ij} = 0$ if $A_{ij} = 1$ and $B_{ij} = 1$, otherwise $M_{ij} = 1$

Link prediction with multiple sources(networks)

- Paths with edges in A:

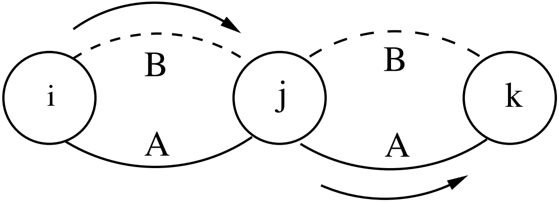
$$i \xrightarrow{A} j \xrightarrow{A} k, \qquad A^2$$

- Paths with edges in B:

$$i \xrightarrow{B} j \xrightarrow{B} k, \qquad B^2$$

- Hybrid paths with edges in A and B:

$$i \xrightarrow{B} j \xrightarrow{A} k, \qquad BA$$



Link prediction with multiple sources(networks)

- In general, utilizing single source and hybrid source paths of maximum length $l_{\max}$ involves terms from

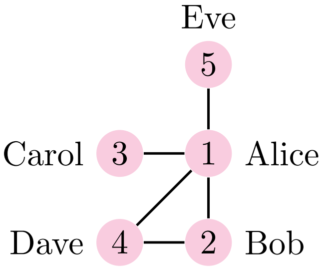
$$(I + A + B + C)^{l_{\max}}$$

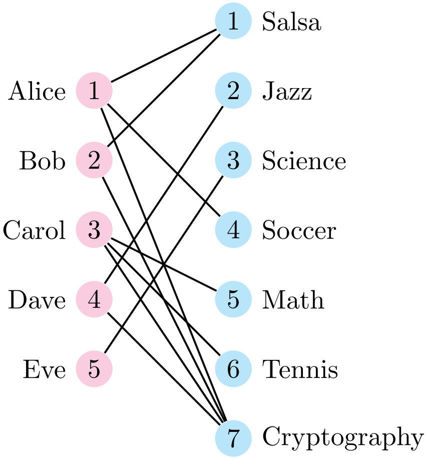
- Compare with truncated Katz
  - Linear combination in terms of $\{A,\ A^2,\ \cdots,\ A^{l_{\max}}\}$
  - Linear combination in terms of

$$\{A,\ B,\ C,\ A^2,\ B^2,\ C^2,\ AB,\ AC,\ BA,\ BC,\ CA,\ CB\} \qquad l_{\max} = 2$$

- Weights learned using "Hierarchical sparsity regularization"
- Utilizing multiple sources can improve link prediction
- Experiments with Arxiv and CiteSeer data
- Sources include:
  1. author $\times$ author collaboration network
  2. paper $\times$ paper citation network
  3. author $\times$ paper network
  4. author $\times$ author networks based on similarities of abstracts, titles and keywords.

Affiliation recommendation





Affiliation recommendation given a social network

A is the affiliation network, S is the social network

- Method 1: Low rank approximation of A
- Method 2: Low rank approximations of the combined matrix

$$C = \begin{bmatrix} \lambda S & A \\ A^T & 0 \end{bmatrix} \approx \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \Lambda \begin{bmatrix} V_1^T & V_2^T \end{bmatrix},$$

  where λ controls amount of influence of the social network.
- Predictions based on $V_1 \Lambda V_2^{\mathrm{T}}$

Affiliation recommendation given a social network

- Method 3: Katz or truncated Katz measure on the combined matrix

$$P_{\text{katz}} = \beta C + \beta^2 C^2 + \cdots$$

$$P_{\text{t-katz}} = \beta C + \beta^2 C^2 + \cdots + \beta^l C^l$$

  Predictions are based on the (1,2) block (corresponding to A in C), e.g.

$$P^{(1,2)}_{\text{t-katz}} = \beta A + \beta^2 \lambda S A + \beta^3 \lambda^2 (S^2 A + A A^{\mathsf{T}} A)$$

  We can use both low rank approach and Katz measure approach.

- Method 4: Common subspace to approximate A and S

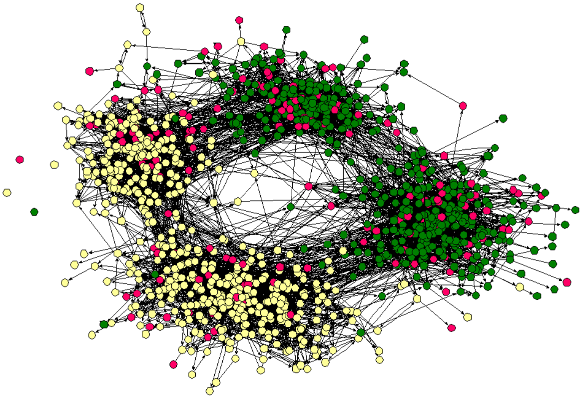
$$A \approx Q D_A V^{\mathsf{T}}, \qquad S \approx Q D_S Q^{\mathsf{T}}$$

  Q spans dominant subspaces of both A and S

Graph Clustering/Community Detection

Graph Partitioning/Clustering

- In many applications, goal is to partition/cluster nodes of a graph:

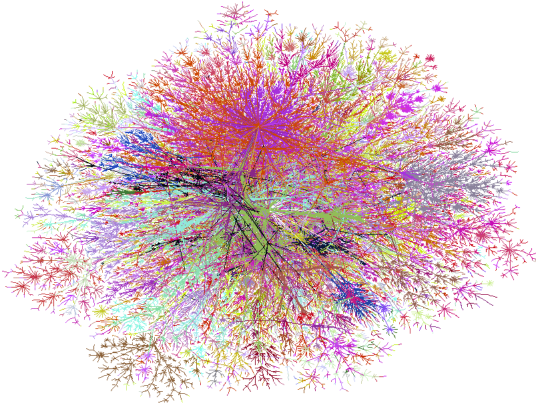


High School Friendship Network

[James Moody. American Journal of Sociology, 2001]

Graph Partitioning/Clustering

- In many applications, goal is to partition/cluster nodes of a graph:



The Internet

[The Internet Mapping Project, Hal Burch and Bill Cheswick, Lumeta Corp, 1999]

Graph clustering

- Cluster vertices $\mathcal{V}$ into c disjoint sets $\mathcal{V}_i$,

$$\mathcal{V} = \cup_{i=1}^{c} \mathcal{V}_i \quad \text{and} \quad \mathcal{V}_i \cap \mathcal{V}_j = \emptyset \qquad i \neq j$$

- Objective functions for clustering

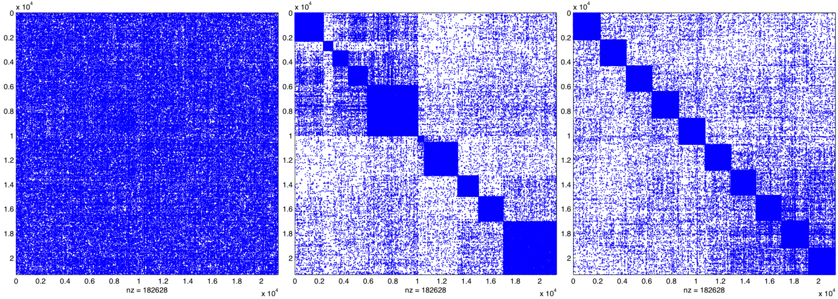
$$\text{RCut} = \min_{\mathcal{V}_1,\ldots,\mathcal{V}_c} \sum_{i=1}^{c} \frac{\text{links}(\mathcal{V}_i, \mathcal{V} n \mathcal{V}_i)}{|\mathcal{V}_i|} \qquad \text{(Chan et al. 1994)}$$

$$\text{NCut} = \min_{\mathcal{V}_1,\ldots,\mathcal{V}_c} \sum_{i=1}^{c} \frac{\text{links}(\mathcal{V}_i, \mathcal{V} n \mathcal{V}_i)}{\text{degree}(\mathcal{V}_i)} \qquad \text{(Shi and Malik 2000)}$$

$$\text{KLObj} = \min_{\mathcal{V}_1,\ldots,\mathcal{V}_c} \sum_{i=1}^{c} \frac{\text{links}(\mathcal{V}_i, \mathcal{V} n \mathcal{V}_i)}{|\mathcal{V}_i|} \quad \text{s.t.} \quad |\mathcal{V}_i| = |\mathcal{V}_j| \qquad \text{(Kernighan and Lin 1970)}$$

- Spectral clustering can be expensive for massive graphs
- Fast clustering algorithms without eigenvector computations
  - GRACLUS (Dhillon et al. 2007) and METIS (Karypis and Kumar 1999)

Clustering example: arXiv data graph **condMat**



# of clust	edges in clust	# of vertices	# of edges	relative error
10 (graclus)	79.8 %	21,363	182,628	45.0
10 (metis)	77.9 %	21,363	182,628	47.0

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1c} \\ \vdots & \ddots & \vdots \\ A_{c1} & \cdots & A_{cc} \end{bmatrix}$$

Motivation: main idea through an example

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$$

Let $\sigma_1(A_{22}) < \sigma_k(A_{11})$ and $\bar{U}_k \bar{\Sigma}_k \bar{V}_k^{\mathsf{T}} \approx A_{11}$

$$A \approx U_k \Sigma_k V_k^{\mathsf{T}} = \begin{bmatrix} \bar{U}_k \bar{\Sigma}_k \bar{V}_k^{\mathsf{T}} & 0 \\ 0 & 0 \end{bmatrix}$$

No information from A_{22} is extracted.
Approximate each block: $\bar{U}_k \bar{\Sigma}_k \bar{V}_k^{\mathsf{T}} \approx A_{11}$ and $\hat{U}_k \hat{\Sigma}_k \hat{V}_k^{\mathsf{T}} \approx A_{22}$

$$A \approx \begin{bmatrix} \bar{U}_k & 0 \\ 0 & \hat{U}_k \end{bmatrix} \begin{bmatrix} \bar{\Sigma}_k & 0 \\ 0 & \hat{\Sigma}_k \end{bmatrix} \begin{bmatrix} \bar{V}_k & 0 \\ 0 & \hat{V}_k \end{bmatrix}^{\mathsf{T}},$$

Main idea continued

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

Compute a low rank approximation of each diagonal block

$$U_i S_{ii} V_i^{\mathsf{T}} \approx A_{ii} \qquad U_i,\ V_i \text{ orthonormal}$$

Approximate now

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \approx \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}^{\mathsf{T}}$$

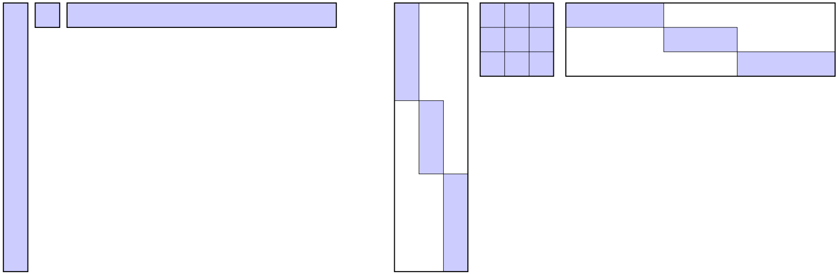
where

$$S_{ij} = U_i^{\mathsf{T}} A_{ij} V_j$$

Low rank vs clustered low-rank

Low rank: $A \approx U \Sigma V^{\mathsf{T}}$

Clustered low rank: $A \approx \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}^{\mathsf{T}}$



- Observe $\mathrm{diag}(U_1, U_2, \cdots, U_c)$ and has the same memory usage as U
- Experiments show that most of U is contained in $\mathrm{diag}(U_1, U_2, \cdots, U_c)$
- Similarly of V and $\mathrm{diag}(V_1, V_2, \cdots, V_c)$

Algorithm: Clustered low rank approximation

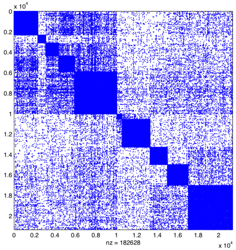
Input: An $m \times m$ adjacency matrix A of a graph, number of clusters c
Output: Clustered low rank approximation of A

1: Cluster the graph into c clusters
2: Compute a low rank approximation of each cluster

$$U_i S_i V_i \approx A_{ii}$$

3: Extend the cluster-wise approximations, into an approximation for the entire matrix A

$$S_{ij} = U_i^{\mathsf{T}} A_{ij} V_j$$



Different low rank approximations

- Deterministic methods — truncated SVD using ARPACK, PROPACK, SVDPACK.
- Stochastic methods

$$Y = A\Omega \qquad \text{or} \qquad Y = (AA^T)^q A\Omega$$

where q is small (1,2 or 3), and Ω is a random matrix (standard Gaussian). An approximation is obtained with

$$A \approx \hat{A} = P_Y A = Y(Y^T Y)^{-1} Y^T A$$

where P_Y is an orthogonal projection onto the range space of Y

Deterministic error bound—preliminaries

For any partitioning

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1c} \\ \vdots & \ddots & \vdots \\ A_{c1} & \cdots & A_{cc} \end{bmatrix}$$

and any $Y^{(i)}$ form

$$Y = \begin{bmatrix} Y^{(1)} & & \\ & \ddots & \\ & & Y^{(c)} \end{bmatrix} \qquad P_Y = \begin{bmatrix} P_{Y^{(1)}} & & \\ & \ddots & \\ & & P_{Y^{(c)}} \end{bmatrix}$$

P_Y and each $P_{Y^{(i)}}$ are orthogonal projectors

Deterministic clustered error bound

The approximation error is given by

$$\|A - \hat{A}\| = \|A - P_Y A\| = \|(I - P_Y)A\|$$

Theorem

$$\|(I - P_Y)A\|_F^2 \leq \sum_{i,j=1}^{c} \|(I - P_{Y^{(i)}})A_{ij}\|_F^2$$

Introduce variables

Assume we have c clusters

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1c} \\ \vdots & \ddots & \vdots \\ A_{c1} & \cdots & A_{cc} \end{bmatrix}$$

Generate

$$Y^{(i)} = A_{ii}\Omega^{(i)}$$
$$Y^{(i)} = (A_{ii}A_{ii}^{\top})^{q}A_{ii}\Omega^{(i)}$$

using random $\Omega^{(i)}$

$$Y = \begin{bmatrix} Y^{(1)} & & \\ & \ddots & \\ & & Y^{(c)} \end{bmatrix} \qquad P_Y = \begin{bmatrix} P_{Y^{(1)}} & & \\ & \ddots & \\ & & P_{Y^{(c)}} \end{bmatrix}$$

Stochastic error bounds in the clustered setting

Theorem

$$\mathbb{E}\|(I - P_Y)A\|_F \leq \sum_{i=1}^{c} \left(1 + \frac{k_i}{p_i + 1}\right)^{1/2} \|\Sigma_2^{(i)}\|_F + \sum_{i,j=1,\ i \neq j}^{c} \|A_{ij}\|_F^2$$

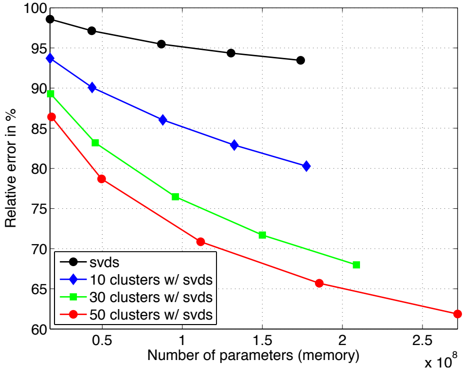
$$\mathbb{E}\|(I - P_Y)A\|_2 \leq \sum_{i=1}^{c} \left(\left(1 + \frac{\sqrt{k_i}}{\sqrt{p_i - 1}}\right)\|\Sigma_2^{(i)}\|_2 + \frac{e\sqrt{k_i + p_i}}{p_i}\|\Sigma_2^{(i)}\|_F\right) + \sum_{i,j=1,\ i \neq j}^{c} \|A_{ij}\|_2^2$$

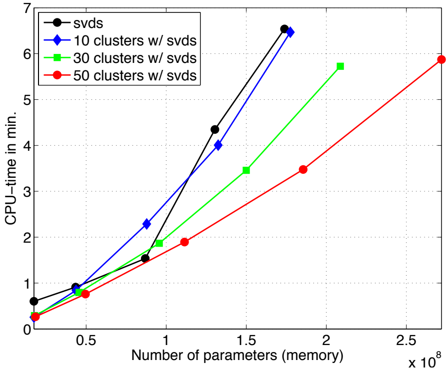
$$A_{ii} \approx U^{(i)}\Sigma^{(i)}(V^{(i)})^\mathsf{T} = U^{(i)} \begin{bmatrix} \Sigma_1^{(i)} & 0 \\ 0 & \Sigma_2^{(i)} \end{bmatrix} \begin{bmatrix} V_1^{(i)} & V_2^{(i)} \end{bmatrix}^\mathsf{T}$$

Experimental Results

Google web matrix—performance comparison

- 434,818 vertices; 3,419,124 edges
- Edges within clusters: 98.2%, 97.8%, 97.6%; graph is highly clusterable
- "Ranks" are $k = 20, 50, 100, 150, 200$.





LiveJournal graph

- 3,828,682 vertices; 65,825,429 edges
- Edges within clusters: 76.9%, 69.3%, 66.3%; graph not highly clusterable
- "Ranks" are $k = 20, 50, 100, 150, 200$.

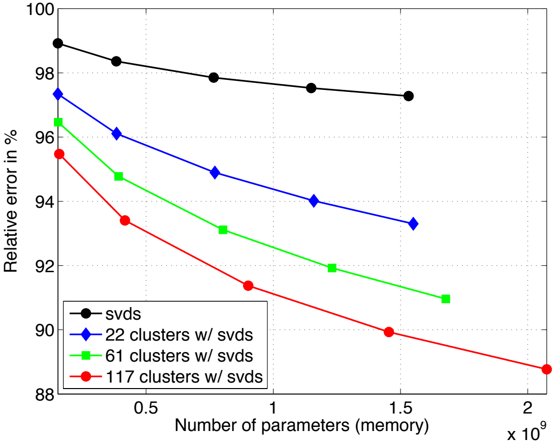
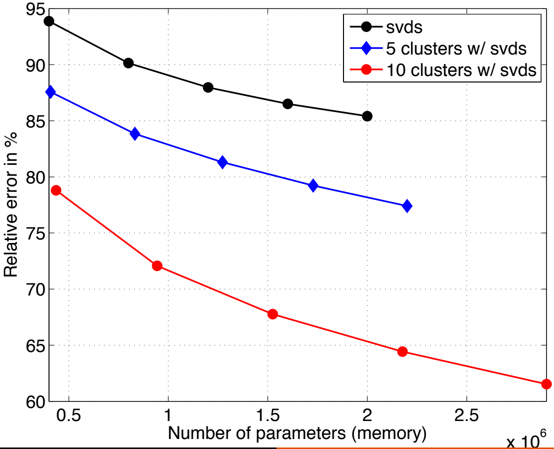


Image segmentation graph

Here we use the normalized adjacency matrix:

- 10,000 vertices; 1,091,910 edges
- Edges within clusters: 77.5%, 70.4%; graph not highly clusterable



Testbed of Social Networks

Network	Date	# nodes	# links	# added links	% added links
Flickr	4/14/2007	1,990,149	41,302,536	–	–
	4/25/2007	1,990,149	42,056,754	754,218	1.8%
	5/6/2007	1,990,149	42,879,714	822,960	1.9%
LiveJournal	02/16/2009	1,770,961	83,663,478	–	–
	03/4/2009	1,770,961	84,413,542	750,064	0.8%
	04/03/2009	1,770,961	85,713,766	1,300,224	1.5%
MySpace	12/11/2008	2,137,264	90,333,122	–	–
	1/11/2009	2,137,264	90,979,264	646,142	0.7%
	2/14/2009	2,137,264	91,648,716	669,452	0.7%

	# clusters	avg size	% intra links	% inter links
Flickr	18	110,563	71.8%	28.2%
LiveJournal	17	106,241	72.5%	27.5%
MySpace	17	125,721	51.9%	48.1%

Link prediction evaluation — have to sample

Introduce the sets

$$\begin{aligned} \mathcal{E} &= \{(i,j) \mid A_{ij} = 1\}, && \text{“old links”} \\ \mathcal{N} &= \{(i,j) \mid A_{ij} = 0 \ \&\ B_{ij} = 1\}, && \text{newly formed links} \\ \mathcal{Z} &= \{(i,j) \mid A_{ij} = 0 \ \&\ B_{ij} = 0\}, && \text{no links between vertices } i \text{ and } j \end{aligned}$$

Concrete example (MySpace): A is 2.1M $\times$ 2.1M matrix with

$$|\mathcal{E}| \approx 0.0020\%, \quad |\mathcal{N}| \approx 1.46 \cdot 10^{-5}\%, \quad |\mathcal{Z}| \approx 99.998\%$$

Training Randomly sample 100,000 links from $\mathcal{N}$ and 500,000 links from $\mathcal{Z}$, true ratio 1 to 6.8M

Testing Randomly sample testing subsets from $\mathcal{N}$ and $\mathcal{Z}$, compute corresponding scores, and predict links on highest scores.

Predictions Generic predictor in factored form

$$P = \bar{V}\bar{D}\bar{V}^{\mathsf{T}}$$

we can compute P_{ij} for any ij

Performance of link predictors

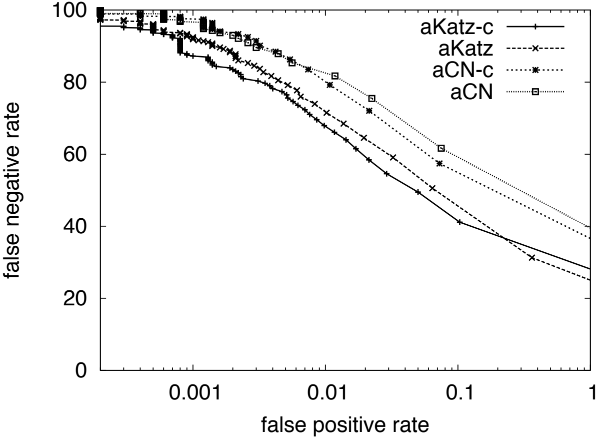
1. Compute scores for 100k positive & 500k negative links (600k total)
2. Sort the scores
3. For a given threshold consider all scores above this threshold as predictions of friendships
4. Plot false positive rate (FPR) vs false negative rate (FNR) for a range of threshold values

$$
\begin{aligned}
FPR &= \frac{\#\text{of incorrectly predicted friend links}}{\#\text{of non-friend pairs}} \\
FNR &= \frac{\#\text{of missed friend links}}{\#\text{of new friend links}}
\end{aligned}
$$

5. Each threshold value gives one point in the plot

Link prediction results on LiveJournal

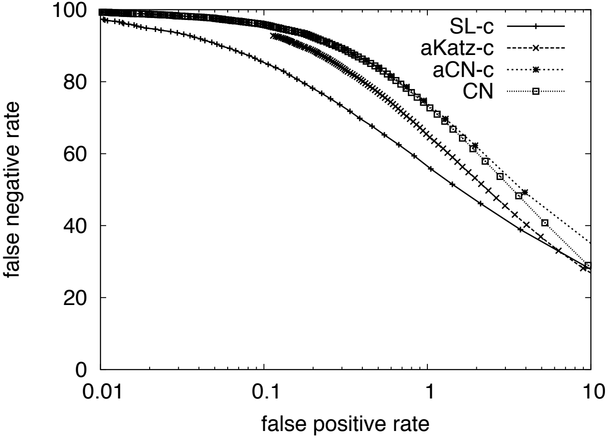
Lower-left better; "-c" indicates clustering is involved



Clustering improves both Katz measures and Common neighbors method

Link prediction results on Flickr

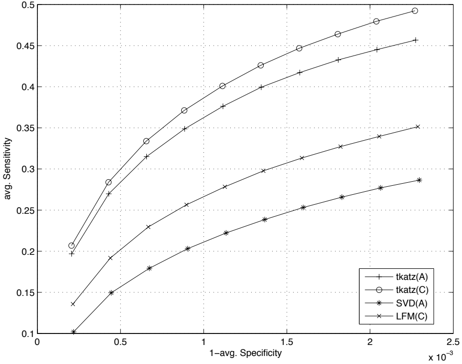
Lower-left better; "-c" indicates clustering is involved



Best method "SL-c" Spectral learning with clustering (not in this talk)

Affiliation recommendation on Youtube data

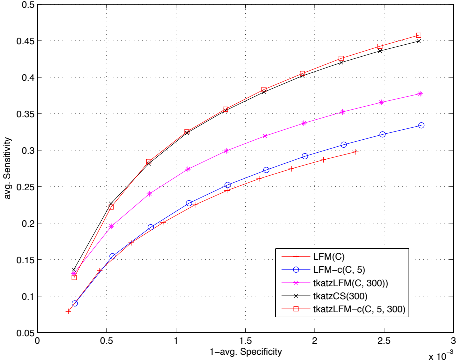
Upper-left is better



Clear improvement in performance by utilizing the social network (for both truncated Katz and latent factors model (LFM)).

Affiliation recommendation on Youtube data

Upper-left is better



Best two methods: truncated Katz measure using Common Subspaces and truncated Katz measure using latent factors model and clustering.

Clustering gives a substantial boost in performance (compare "pink" with "red square")

Summary and Conclusions

- Clustered low rank approximation
  - Improved quality of approximation with identical memory consumption faster than state-of-the-art (PROPACK, SVDPACK, ARPACK). Note: clustering time included.
- Prediction methods
  - Katz and CN using graph embedding
  - Generalized truncated Katz
- Utilizing multiple sources improves link prediction
- Utilizing social network improves affiliation recommendation
- Improved and scalable computation of proximity measures & link predictions with clustered low rank approximation
- Fruitful to combine clustering structure with low rank approximation framework

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[3] V. Vasuki, N. Natarajan, Z. Lu B. Savas and I. S. Dhillon. *Scalable Affiliation Recommendation using Auxiliary Networks*. Submitted for publication, 2010.

[4] Z. Lu, B. Savas, W. Tang, and I. S. Dhillon. *Supervised link prediction using multiple sources*. Submitted for publication, 2010.