Inverse Eigenvalue Problems in Wireless Communications

Inderjit S. Dhillon
Robert W. Heath Jr.
Mátyás Sustik
Joel A. Tropp

The University of Texas at Austin

Thomas Strohmer

The University of California at Davis

Introduction

- Matrix construction problems arise in theory of wireless communication
- Many papers have appeared in *IEEE Trans. on Information Theory*
- We view these constructions as *inverse eigenvalue problems*
  - Provides new insights
  - Suggests new tools for solution
  - Offers new and interesting inverse eigenvalue problems

References: [Rupf-Massey 1994; Vishwanath-Anantharam 1999; Ulukus-Yates 2001; Rose 2001; Viswanath-Anantharam 2002; Anigstein-Anantharam 2003; . . .]

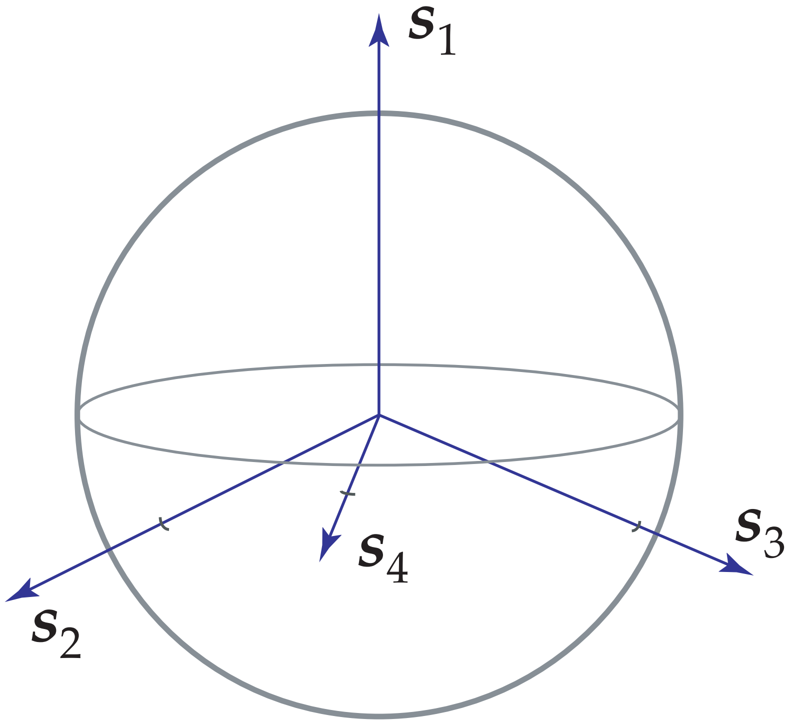
Code-Division Multiple Access (CDMA)

- A CDMA system allows many users to share a wireless channel
- Channel is modeled as a vector space of dimension d
- Each of N users receives a unit-norm signature vector $\boldsymbol{s}_k$ $(N > d)$
- Each user's information is encoded in a complex number b_k
- In each transmission interval, a user sends $b_k\,\boldsymbol{s}_k$
- Each user may have a different power level w_k
- Base station receives superposition $\sum_{k=1}^{N} b_k\,\sqrt{w}_k\,\boldsymbol{s}_k + \boldsymbol{v}$, where $\boldsymbol{v}$ is additive noise
- The base station must extract all b_k from the d-dimensional noisy observation

Reference: [Viterbi 1995]

Example

- Intuition: the signature vectors should be well separated for the system to perform well



$$\frac{1}{3}\begin{bmatrix} 0 & 0 & \sqrt{6} & -\sqrt{6} \\ 0 & 2\sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 3 & -1 & -1 & -1 \end{bmatrix}$$

Optimal CDMA Signatures

- For clarity, suppose the noise is a white, Gaussian random process
- Form the weighted signature matrix

$$X = \left[\sqrt{w_1}\, \boldsymbol{s}_1 \quad \sqrt{w_2}\, \boldsymbol{s}_2 \quad \dots \quad \sqrt{w_N}\, \boldsymbol{s}_N \right]$$

- One performance measure is *total weighted squared correlation* (TWSC)

$$\mathrm{TWSC}(X) \stackrel{\text{def}}{=} \|X^* X\|_{\mathrm{F}}^2 = \sum w_j\, w_k\, |\langle \boldsymbol{s}_j, \boldsymbol{s}_k \rangle|^2$$

- Minimizing TWSC is (often) equivalent to finding X for which

$$XX^* = \frac{\sum w_k}{d}\, \mathrm{I}_d \qquad \text{and} \qquad \mathrm{diag}\,(X^* X) = (w_1, \dots, w_N)$$

- Thus X is row-orthogonal with specified column norms

References: [Rupf-Massey 1994; Vishwanath-Anantharam 1999, 2002]

Connection with Tight Frames

- An *α-tight frame* is a collection $\{\boldsymbol{x}_k\}$ of N vectors in $\mathbb{C}^d$ such that

$$\sum_{k=1}^{N} |\langle \boldsymbol{y}, \boldsymbol{x}_k \rangle|^2 = \alpha \, \|\boldsymbol{y}\|_2^2 \qquad \text{for all } \boldsymbol{y} \text{ in } \mathbb{C}^d$$

- α-tight frames generalize orthonormal systems

- Designing tight frames with specified norms $\equiv$ Designing optimal CDMA signatures under white noise

- Tight frames also arise in signal processing, harmonic analysis, physics, ...

Spectral Properties of Tight Frames

- The *frame synthesis matrix* is defined as $X \stackrel{\text{def}}{=} \begin{bmatrix} \boldsymbol{x}_1 & \dots & \boldsymbol{x}_N \end{bmatrix}$

- Observe that the tight frame condition can be written

$$\frac{\boldsymbol{y}^*(XX^*)\boldsymbol{y}}{\boldsymbol{y}^*\boldsymbol{y}} = \alpha \qquad \text{for all } \boldsymbol{y} \text{ in } \mathbb{C}^d$$

- Four equivalent definitions of a tight frame:
  - The rows of X are orthogonal
  - The d singular values of X are identical
  - The d non-zero eigenvalues of X^*X are identical
  - The Gram matrix X^*X is a scaled rank-d orthogonal projector

Structural Constraints on Frame Vectors

- Prescribed Euclidean norms
  - This is the CDMA signature design problem
- Low peak-to-average-power ratio
  - Components of each vector should have similar moduli
- Low cross-correlations $|\langle \boldsymbol{x}_j, \boldsymbol{x}_k \rangle|$ between each pair
  - Vectors in tight frames can have large pairwise correlations
  - Preferable for all vectors to be well separated
- Components drawn from a finite alphabet
  - Fundamental problem in communications engineering
  - One common alphabet is $\mathscr{A} = \{(\pm 1 \pm \mathrm{i})/\sqrt{2}\}$
- . . .
- . . .

Inverse Singular Value Problems

- Let $\mathscr{S}$ be a collection of "structured" $d \times N$ matrices
- Let $\mathscr{X}$ be the collection of $d \times N$ matrices with singular values $\sigma_1, \ldots, \sigma_d$
- Find a matrix in the intersection of $\mathscr{S}$ and $\mathscr{X}$
- If problem is not soluble, find a matrix in $\mathscr{S}$ that is closest to $\mathscr{X}$ with respect to some norm
- General numerical approaches are available
- Inverse eigenvalue problems defined similarly for the $N \times N$ Gram matrix

References: [Chu 1998, Chu-Golub 2002]

Algorithms

Finite-step methods

- Useful for simple structural constraints
- Fast and easy to implement
- Always succeed

Alternating projection methods

- Good for more complicated structural constraints
- Slow but easy to implement
- May fail

Projected gradient or coordinate-free Newton methods

- Difficult to develop; not good at repeated eigenvalues
- Fairly fast but hard to implement
- May fail

Finite-Step Methods

- Goal: construct tight frame X with squared column norms $w_1, \ldots, w_N$
- Equivalent to Schur-Horn Inverse Eigenvalue Problem
  - Gram matrix X^*X has diagonal $w_1, \ldots, w_N$
  - Gram matrix has d non-zero eigenvalues, all equal to $\sum w_k / d$
  - Diagonal must majorize eigenvalues: $0 \leq w_j \leq \sum w_k / d$ for all j

Basic Idea

- Start with diagonal matrix of eigenvalues
- Apply sequence of $(N-1)$ plane rotations [Chan-Li 1983]

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0.4000 & 0.4323 & -0.2449 \\ 0.4323 & 0.7000 & 0.1732 \\ -0.2449 & 0.1732 & 0.9000 \end{bmatrix}$$

- Extract the frame X with rank-revealing QR [Golub-van Loan 1996]

Finite-Step Methods

Equal Column Norms

- Start with arbitary Hermitian matrix whose trace is $\sum w_k$
- Apply $(N-1)$ plane rotations [Bendel-Mickey 1978, GvL 1996]

$$\begin{bmatrix} 0.6911 & 1.1008 & -1.0501 \\ 1.1008 & 1.8318 & -0.9213 \\ -1.0501 & -0.9213 & -0.5229 \end{bmatrix} \longmapsto \begin{bmatrix} 0.6667 & -1.4933 & -0.5223 \\ -1.4933 & 0.6667 & 1.4308 \\ -0.5223 & 1.4308 & 0.6667 \end{bmatrix}$$

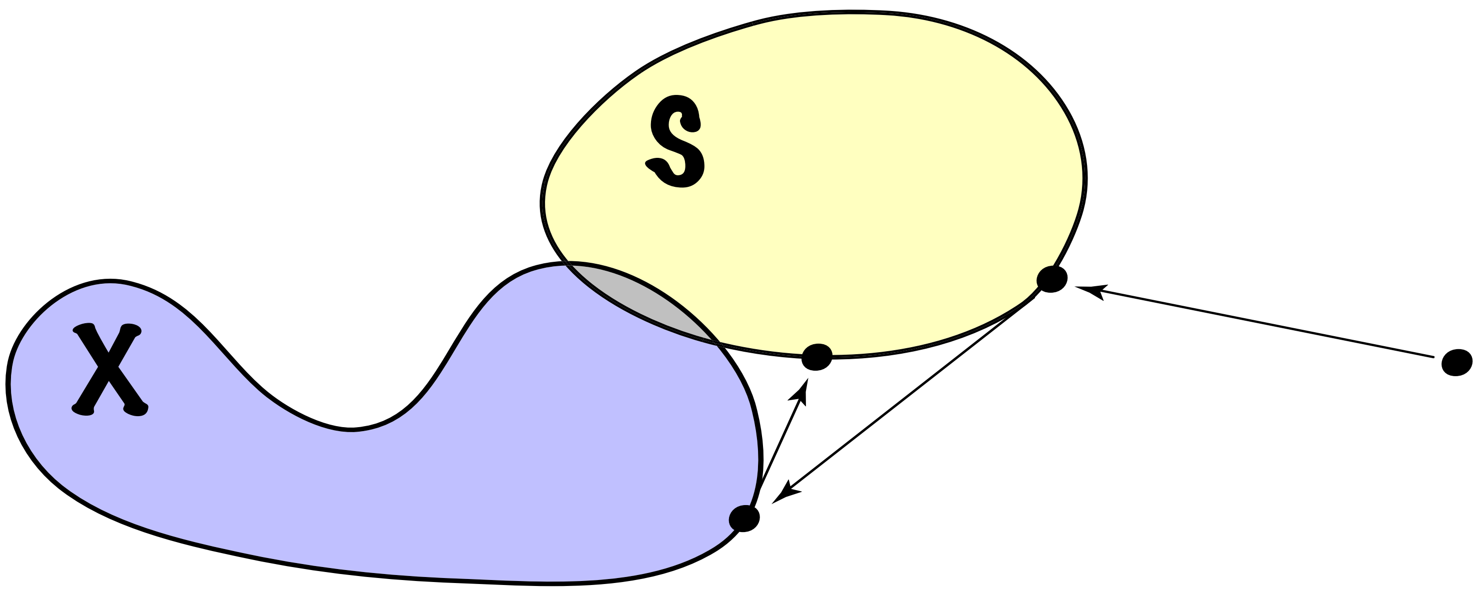
- Extract the frame X with rank-revealing QR factorization

One-Sided Methods

- Can use Davies-Higham method [2000] to construct tight frames with equal column norms directly
- We have extended Chan-Li to construct tight frames with arbitrary column norms directly [TDH 2003, DHSuT 2003]

Alternating Projections

- Let $\mathscr{S}$ be the collection of matrices that satisfy the structural constraint
- Let $\mathscr{X}$ be the collection of α-tight frames
- Begin with an arbitrary matrix
- Find the nearest matrix that satisfies the structural constraint
- Find the nearest matrix that satisfies the spectral constraint. . .



Literature on Alternating Projections

Theory

- Subspaces [J. Neumann 1933; Diliberto-Straus 1951; Wiener 1955; . . .]
- Convex sets [Cheney-Goldstein 1959]
- Descent algorithms [Zangwill 1969; R. Meyer 1976; Fiorot-Huard 1979]
- Corrected [Dykstra 1983; Boyle-Dykstra 1985; Han 1987]
- Information divergences [Csiszár-Tusnády 1984]
- Recent surveys [Bauschke-Borwein 1996; Deutsch 2001]

Practice

- Signal recovery and restoration [Landau-Miranker 1961; Gerchberg 1973; Youla-Webb 1982; Cadzow 1988; Donoho-Stark 1989; . . .]
- Schur-Horn IEP [Chu 1996]
- Nearest symmetric diagonally dominant matrix [Raydan-Tarazaga 2000]
- Nearest correlation matrix [Higham 2002]

Nearest Frames & Gram Matrices

- To implement the alternating projection, one must compute the tight frame or tight frame Gram matrix nearest a given matrix
- For analytic simplicity, we use the Frobenius norm

Theorem 1. *Suppose that Z has polar decomposition $R\Theta$. The matrix Θ is a tight frame nearest to Z. If Z has full rank, the nearest matrix is unique.*

Theorem 2. *Let Z be a Hermitian matrix, and let the columns of U be an orthonormal basis for an eigenspace associated with the d algebraically largest eigenvalues. Then UU^* is a rank-d orthogonal projector closest to Z. The nearest projector is unique if and only if $\lambda_d(Z) > \lambda_{d+1}(Z)$.*

References: [Horn-Johnson 1985]

Nearest Matrix with Specified Column Norms

- Consider the structural constraint set

$$\mathscr{S} = \{S \in \mathbb{C}^{d \times N} : \|\boldsymbol{s}_k\|_2^2 = w_k\}$$

Proposition 1. *Let Z be an arbitrary matrix. A matrix in $\mathscr{S}$ is closest to Z if and only if*

$$\boldsymbol{s}_k = \begin{cases} w_k \, \boldsymbol{z}_k / \|\boldsymbol{z}_k\|_2 & \text{for } \boldsymbol{z}_k \neq \mathbf{0} \quad \text{and} \\ w_k \, \boldsymbol{u}_k & \text{for } \boldsymbol{z}_k = \mathbf{0}, \end{cases}$$

where $\boldsymbol{u}_k$ is an arbitrary unit vector. If the columns of Z are all non-zero, then the solution to the nearness problem is unique.

Convergence for Fixed Column Norms

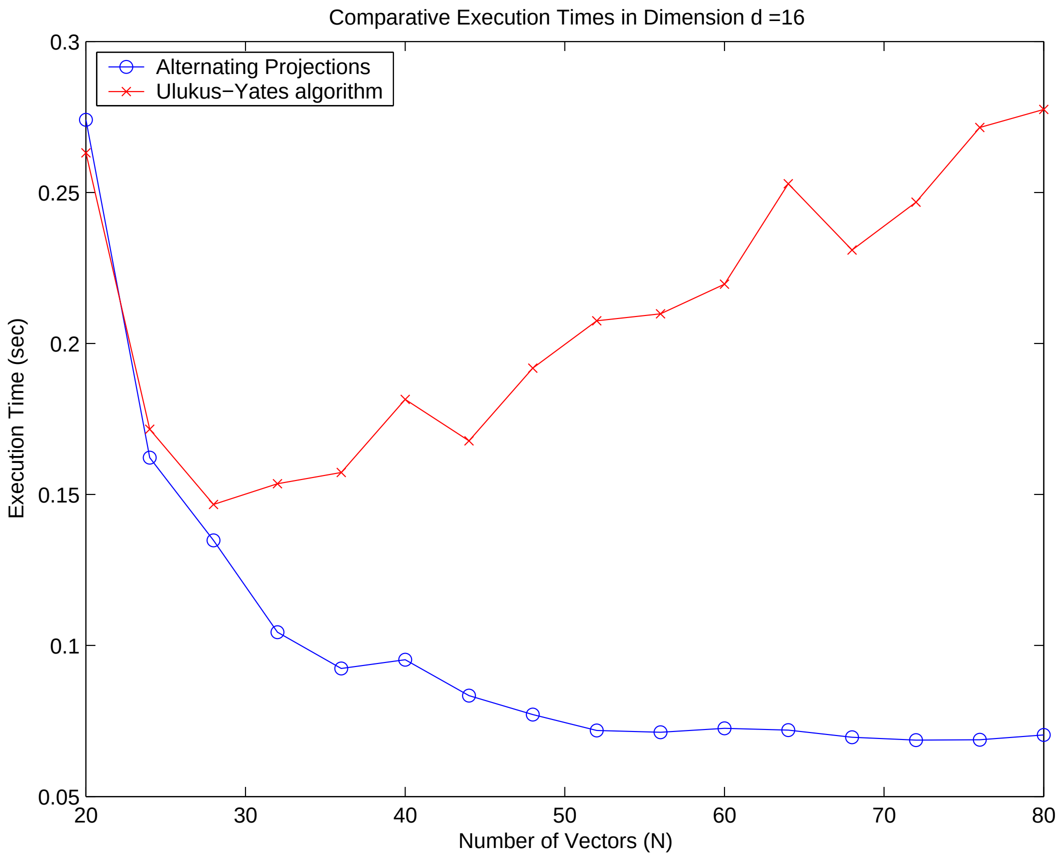
Theorem 3. [THSt 2003] *Suppose that S_0 has full rank and non-zero columns. Perform an alternating projection between $\mathscr{S}$ and $\mathscr{X}$. The sequence of iterates either converges in norm to a full-rank fixed point of the algorithm or it has a continuum of accumulation points that are all full-rank fixed points of the algorithm.*

Theorem 4. [THSt 2003] *The full-rank stationary points of the alternating projection between $\mathscr{S}$ and $\mathscr{X}$ are precisely the full-rank matrices in $\mathscr{S}$ whose columns are all eigenvectors of SS^*. That is, $SS^*S = S\Lambda$ where Λ is diagonal and positive.*

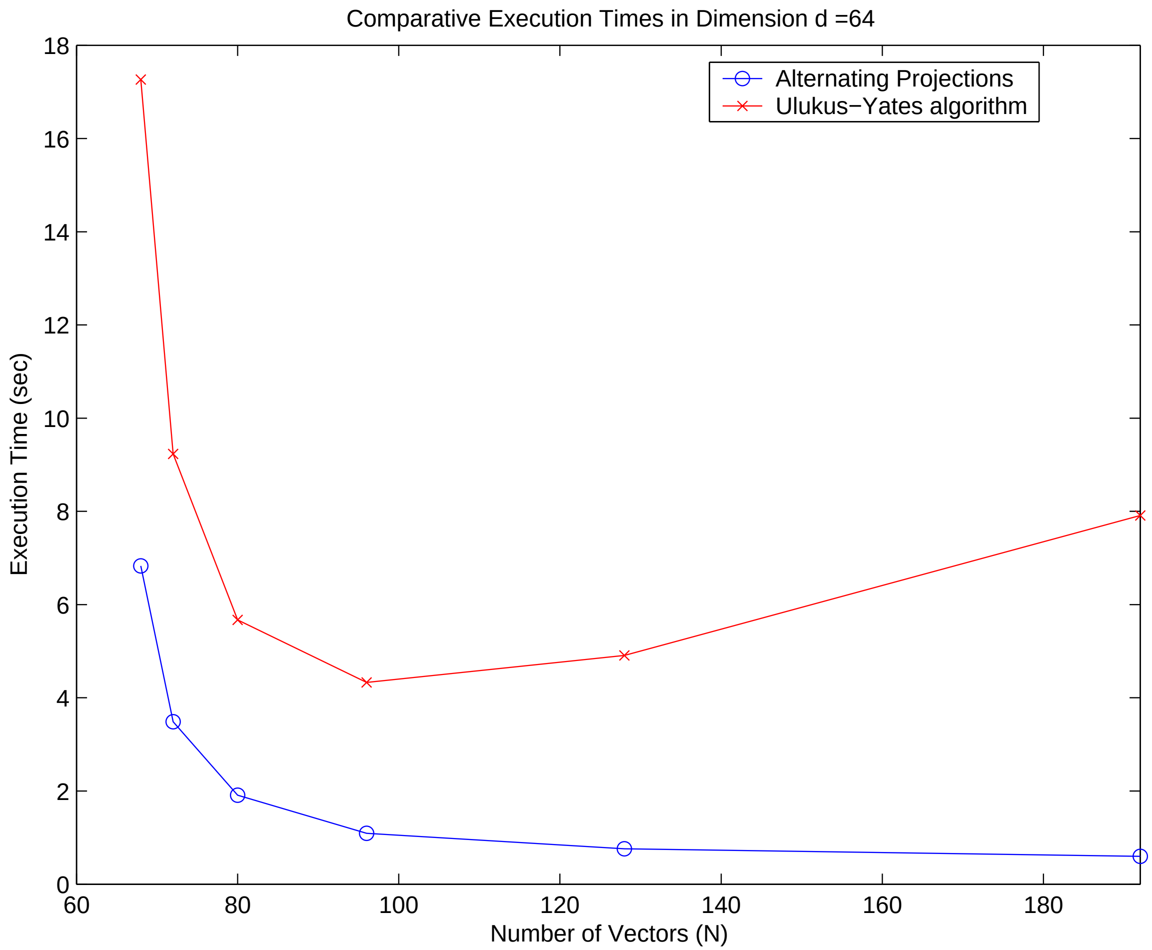
- Each fixed point may be identified as union of tight frames for mutually orthogonal subspaces of $\mathbb{C}^d$ [Ulukus-Yates 2001; Benedetto-Fickus 2002; Anigstein-Anantharam 2003]

Alternating Projections vs. Ulukus-Yates

- Other algorithms have been proposed for constructing tight frames with specified column norms, eg. [Ulukus-Yates 2001]



Alternating Projections vs. Ulukus-Yates



Peak-to-Average-Power Ratio

- In communications applications, it is practical for the vectors to have components with similar moduli
- Define the *peak-to-average-power ratio* of a vector $\boldsymbol{v}$ in $\mathbb{C}^d$ to be

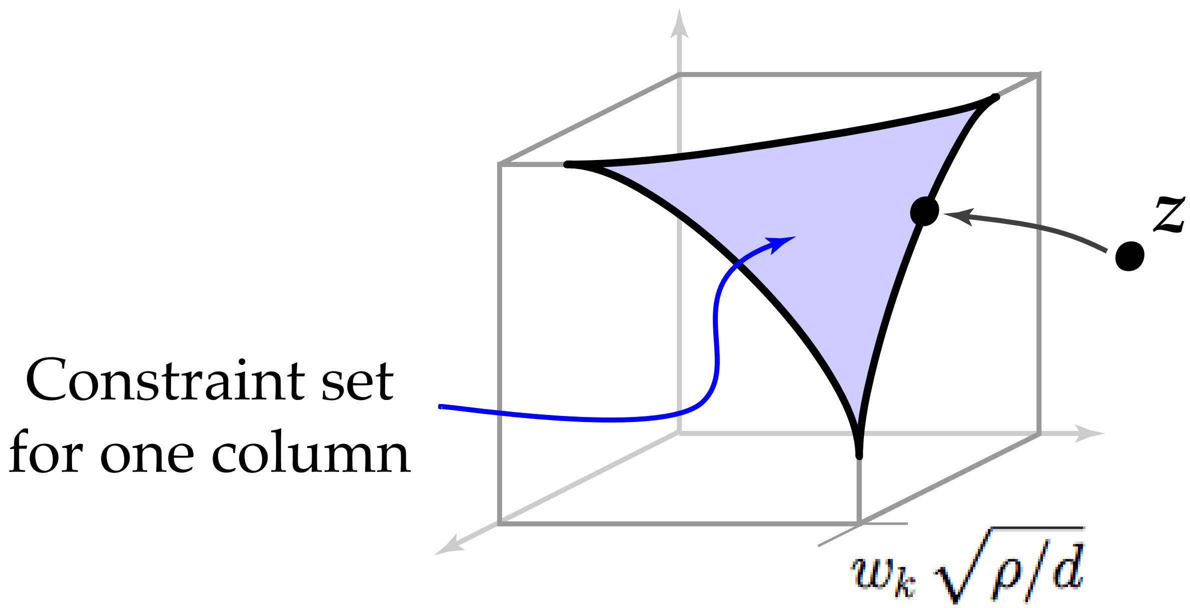
$$\mathrm{PAR}(\boldsymbol{v}) \stackrel{\text{def}}{=} \frac{\max_j |v_j|^2}{\sum_j |v_j|^2 / d}$$

- Note that $1 \le \mathrm{PAR}(\boldsymbol{v}) \le d$
  - The lower extreme corresponds to equal-modulus vectors
  - The upper bound occurs only for scaled canonical basis vectors

The PAR Constraint

- Let ρ be the maximum allowable PAR
- Suppose the frame vectors have norms $w_1, \dots, w_N$
- The constraint set becomes

$$\mathscr{S} = \{S \in \mathbb{C}^{d \times N} : \mathrm{PAR}(\boldsymbol{s}_k) \leq \rho \quad \text{and} \quad \|\boldsymbol{s}_k\|_2^2 = w_k\}$$



Optimal Grassmannian Frames

- An interesting (and difficult) problem is to construct a unit-norm tight frame with minimally correlated vectors
- For any $d \times N$ matrix Z with unit-norm columns

$$\max_{m \neq n} |\langle \boldsymbol{z}_j, \boldsymbol{z}_k \rangle| \geq \sqrt{\frac{N-d}{d\,(N-1)}}.$$

- The matrices that meet the bound are called *optimal Grassmannian (tight) frames*
- Each pair of columns has identical cross-correlation $|\langle \boldsymbol{z}_j, \boldsymbol{z}_k \rangle|$
- They do not exist for most combinations of d and N
- Closely related to "packings in Grassmannian manifolds"

References: [Conway-Hardin-Sloane 1996; StH 2003, SuTDH 2003]

Constructing Optimal Grassmannian Frames

- Let $\mu = \sqrt{(N-d)/(d(N-1))}$
- Consider the constraint sets

$$\mathscr{S} = \{S \in \mathbb{C}^{N \times N} : S = S^*; \quad \operatorname{diag} S = \mathbf{e}; \quad |s_{jk}| \leq \mu\}$$

$$\mathscr{X} = \{X \in \mathbb{C}^{N \times N} : X = X^*; \quad \boldsymbol{\lambda}(X) = (\underbrace{N/d, \dots, N/d}_{d}, 0, \dots, 0)\}$$

- Any matrix in $\mathscr{S} \cap \mathscr{X}$ is an optimal Grassmannian frame
- Empirically, an alternating projection between $\mathscr{S}$ and $\mathscr{X}$ appears to find optimal Grassmannian frames when they exist

Reference: [TDHSt 2003, DHSST 2003]

Tight Frames vs. Grassmannian Frames

Tight frame:

$$X = \begin{bmatrix} -0.6669 & -0.3972 & 0.9829 & 0.1984 & 0.5164 & -0.3540 \\ 0.6106 & 0.4999 & -0.0761 & 0.5205 & 0.4776 & -0.9341 \\ 0.4272 & -0.7696 & 0.1676 & 0.8305 & -0.7108 & -0.0470 \end{bmatrix}$$

$$X^*X = \begin{bmatrix} 1.0000 & 0.2414 & -0.6303 & 0.5402 & -0.3564 & -0.3543 \\ 0.2414 & 1.0000 & -0.5575 & -0.4578 & 0.5807 & -0.2902 \\ -0.6303 & -0.5575 & 1.0000 & 0.2947 & 0.3521 & -0.2847 \\ 0.5402 & -0.4578 & 0.2947 & 1.0000 & -0.2392 & -0.5954 \\ -0.3564 & 0.5807 & 0.3521 & -0.2392 & 1.0000 & -0.5955 \\ -0.3543 & -0.2902 & -0.2847 & -0.5954 & -0.5955 & 1.0000 \end{bmatrix}$$

Grassmannian frame:

$$X = \begin{bmatrix} -0.1619 & -0.6806 & 0.1696 & 0.3635 & -0.4757 & 0.3511 \\ 0.6509 & 0.1877 & -0.4726 & 0.2428 & -0.5067 & -0.0456 \\ -0.2239 & 0.0391 & -0.4978 & -0.5558 & -0.1302 & 0.6121 \end{bmatrix}$$

$$X^*X = \begin{bmatrix} 1.0000 & 0.4472 & -0.4472 & 0.4472 & -0.4472 & -0.4472 \\ 0.4472 & 1.0000 & -0.4472 & -0.4472 & 0.4472 & -0.4472 \\ -0.4472 & -0.4472 & 1.0000 & 0.4472 & 0.4472 & -0.4472 \\ 0.4472 & -0.4472 & 0.4472 & 1.0000 & -0.4472 & -0.4472 \\ -0.4472 & 0.4472 & 0.4472 & -0.4472 & 1.0000 & -0.4472 \\ -0.4472 & -0.4472 & -0.4472 & -0.4472 & -0.4472 & 1.0000 \end{bmatrix}$$

Conclusions

- Wireless is a timely application
- It yields inverse eigenvalue problems and matrix nearness problems
- Tight frames generalize orthogonal bases and have other applications
- The linear algebra community may be able to contribute significantly

Papers

- [THSt] "Inverse eigenvalue problems, alternating minimization and optimal CDMA signature sequences." *Proceedings of IEEE International Symposium on Information Theory*. July 2003.
- [TDHSt] "CDMA signature sequences with low peak-to-average ratio via alternating minimization." To appear at Asilomar, November 2003.
- [TDH] "Finite-step algorithms for constructing optimal CDMA signature sequences." Submitted, April 2003.
- [DHSuT] "Generalized finite algorithms for constructing Hermitian matrices with prescribed diagonal and spectrum."
- [TDHSt] "An alternating projection method for designing structured tight frames." In preparation.
- [SuTDH] "Necessary conditions for existence of optimal Grassmannian frames." In preparation.
- [DHSST] "Grassmannian packings via alternating projections." In preparation.

For More Information. . .

- Inderjit S. Dhillon <inderjit@cs.utexas.edu>
- Robert W. Heath Jr. <rheath@ece.utexas.edu>
- Thomas Strohmer <strohmer@math.ucdavis.edu>
- Mátyás Sustik <sustik@cs.utexas.edu>
- Joel A. Tropp <jtropp@ices.utexas.edu>