CS 383C/CAM 383C/M 383E **Homework 4** **Due: Nov. 18, 2002**

1. (Orthogonalization–20pts) Let V be the 30×20 matrix defined by $v_{ij} = p_j^{i-1}, i = 1, \ldots, 30, j = 1, \ldots, 20$, where $p_j = j/20$. Matrices with entries of the form p_j^{i-1} are called *Vandermonde* matrices. According to MATLAB, $\kappa_2(V) \approx 3 \times 10^{13}$, indicating that the columns of V are nearly dependent. Orthonormalize the columns of V by: (i) Classical Gram-Schmidt, (ii) Modified Gram-Schmidt, (iii) QR by Householder reflections and (iv) QR by Givens rotations.

   Present your results in a tabular format showing (i) $\|I_{30} - \hat{Q}^T\hat{Q}\|_2$ and (ii) $\|V - \hat{Q}\hat{R}\|_2$, where $\hat{Q}$ & $\hat{R}$ denote the computed quantities. Also submit your MATLAB code that does the orthonormalization. Do not use the MATLAB function `qr` to perform the decomposition for you. Comment on the accuracy of solutions obtained.

   NOTE: Please do not include printouts of the matrix V or its QR decomposition with your submission.

2. (Householder Reflectors–10pts) Householder Reflectors are of the form $H = I - 2uu^T, \|u\|_2 = 1$. Assume H is $m \times m$.

   a Prove that H is symmetric and orthogonal.

   b What are the eigenvalues of H?

   c What is the determinant of H? Is it $+1$ or -1?

   d How many flops (additions + subtractions + multiplications) are needed to apply H to a vector x, i.e. for Hx?

   e How many flops for $H \cdot A$, where A is $m \times n$?

   f How many flops does it take to reduce matrix A to upper triangular form R by Householder Reflections? Assume A is $m \times n$, $m \geq n$ and R is $n \times n$.

3. (Householder Reflectors versus Givens rotations–5pts) Do Question 3.8 from the Textbook.

4. (Singular Value Decomposition–10pts) Let A be an $m \times n$ real matrix, $m \geq n$, and let $A = U\Sigma V^T$ be its Singular Value Decomposition (SVD), $\Sigma = diag(\sigma_1, \sigma_2, \ldots, \sigma_n)$.

   a What is $\|A\|_F$, the Frobenius norm of A? Write $\|A\|_F$ explicitly in terms of elements of A?

   b What is $trace(A)$? Again, write $trace(A)$ as an explicit function of elements of A.

   c Prove that $\|A\|_F^2 = trace(A^TA)$.

   d Prove that $\|A\|_F^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2$. (Use the fact that $trace(B) = trace(X^{-1}BX)$ for any nonsingular X.)

   e Prove that $A_k = \sum_i \sigma_i u_i v_i^T$ minimizes $\|A - B\|_F^2$ over all rank k matrices B, where B is $m \times n$.