CS 383C/CAM 383C/M 383E **Homework 3** **Due: Oct 12, 1999**

1. (Gram-Schmidt Process) Let

$$v_1 = \begin{bmatrix} 1 \\ \varepsilon \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ \varepsilon \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \varepsilon \end{bmatrix},$$

   and ε be such that $fl(1+\varepsilon^2) = 1$.

   a Apply Classical Gram-Schmidt and show that the computed vectors are not numerically orthogonal, i.e., computed vectors have dot products much larger than ε.

   b Apply Modified Gram-Schmidt and show that the computed vectors are numerically orthogonal, i.e., computed vectors have dot products $= O(\varepsilon)$.

2. (QR Decomposition) Do Question 3.2 from the Textbook.

3. (Householder Reflectors) Householder Reflectors are of the form $H = I - 2uu^T, \|u\|_2 = 1$. Assume H is $m \times m$.

   a Prove that H is symmetric and orthogonal.

   b What are the eigenvalues of H?

   c What is the determinant of H? Is it $+1$ or -1?

   d How many flops (additions + subtractions + multiplications + additions) are needed to apply H to a vector x, i.e. for Hx?

   e How many flops for $H \cdot A$, where A is $m \times n$?

   f How many flops does it take to reduce matrix A to upper triangular form R by Householder Reflections? Assume A is $m \times n$, $m \geq n$ and R is $n \times n$.

4. (Householder Reflectors versus Givens rotations) Do Question 3.8 from the Textbook.

5. (Singular Value Decomposition) Let A be an $m \times n$ real matrix, $m \geq n$, and let $A = U\Sigma V^T$ be its Singular Value Decomposition (SVD), $\Sigma = diag(\sigma_1, \sigma_2, \ldots, \sigma_n)$.

   a What is $\|A\|_F$, the Frobenius norm of A? Write $\|A\|_F$ explicitly in terms of elements of A?

   b What is $trace(A)$? Again, write $trace(A)$ as an explicit function of elements of A.

   c Prove that $\|A\|_F^2 = trace(A^T A)$.

   d Prove that $\|A\|_F^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2$. (Use the fact that $trace(B) = trace(X^{-1}BX)$ for any nonsingular X.)

   e Prove that $A_k = \sum_i \sigma_i u_i v_i^T$ minimizes $\|A - B\|_F^2$ over all rank k matrices B, where B is $m \times n$.