CS 383C/CAM 383C/M 383E **Midterm Exam** **Date: 4 November 1999**

In the following questions, if some aspect is not clear, make reasonable assumptions. **WRITE DOWN** any assumptions you make.

1. (10 points) True or False? If False, give a counter-example.

   a $\|A+B\| \leq \|A\| + \|B\|$ for any matrix norm.

   b Given the $m \times n$ matrix A with $m \geq n$ and $rank(A) = n$, then $Ax = 0 \Rightarrow x = 0$.

   c If Gaussian Elimination with partial pivoting returns inaccurate answers to $Ax = b$, then A must be ill-conditioned.

   d All eigenvalues of a real square matrix are real.

   e If A is orthogonal, $Ax = b$ can be solved for x in $O(n^2)$ time.

2. (10 points)

   a Let x and y be any two vectors. Prove $\|xy^T\|_2 = \|x\|_2 \cdot \|y\|_2$

   b Suppose you are given an $n \times n$ matrix A, and n-dimensional vectors x and b. Define $r = b - Ax$. Give a δA such that $(A + \delta A)x = b$, and $\|\delta A\|_2 = \|r\|_2 / \|x\|_2$.

3. (10 points) Given $A \in \mathcal{R}^{m \times n}$, $m \geq n$.

   a State (without proof) the singular value decomposition(SVD) of A. What are the singular values, and left and right singular vectors?

   b In terms of the singular vectors of A, describe the following linear spaces: (i) column space of A, (ii) row space of A, (iii) null space of A, (iv) null space of A^T.

   c What is the relationship between the dimensions of the above 4 spaces?

   d Compute the SVD of $A = \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$, and give the corresponding 4 linear spaces listed in b.

4. (10 points)

   a Assume the $n \times n$ real matrix A has real eigenvalues and real eigenvectors. The Schur Decomposition Theorem states that: Given A, there exists a real orthogonal matrix Q and an upper triangular matrix T such that $Q^T A Q = T$. Prove this theorem.

   NOTE: A may be nonsymmetric, and the diagonal entries of T equal the eigenvalues of A.

   HINT: Use induction.

   b Give the eigenvalue decompositions and Schur decompositions of:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 5 & -3 \\ -1 & 3 \end{bmatrix}.$$