A Certified Denotational Abstract Interpreter

(Proof Pearl)

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Static Analysis

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- to prove automatically restricted properties
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But such tools are very complex softwares

- How to establish the soundness of these implementations ?

Certified Static Analysis

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A simple idea:

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Program and prove your analysis in the same language !

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Which language ?

Certified Static Analysis

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Program and prove your analysis in the same language !

Which language ?

Program...

Prove...

Extract...

Execute !

This work

We study a small abstract interpreter

- following Cousot’s lecture notes
- represents an embryo of the Astrée analyser

Challenges

- be able to follow the textbook approach without remodeling the algorithms and the proofs
- first machine-checked instance of the motto
  « my abstract interpreter is correct by construction »

Language Syntax

```
Inductive stmt :=
   Assign            (x:var) (e:expr)
 | Skip
 | Assert            (t:test)
 | If          (t:test) (b1 b2:stmt)
 | While           (t:test) (stmt)
 | Seq (i1 i2:stmt).
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Instructions are labelled (program points)

Language Syntax

```
Definition var := word.
Definition pp := word.
Inductive op := Add | Sub | Mult.
Inductive expr :=
   Const (n:Z)
 | Unknown
 | Var (x:var)
 | Numop (o:op) (e1 e2:expr).
Inductive comp := Eq | Lt.
Inductive test :=
 | Numcomp (c:comp) (e1 e2:expr)
 | Not (t:test)
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Record program := {
  p_stmt: stmt;
  p_end: pp;
  vars: list var
}.
```

Language Syntax

binary numbers with at most 32 bits (useful to prove termination)

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main statement

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last label

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variable declaration

Language Semantics

Semantic Domains

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Structural Operational Semantics

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Inductive sos (p:program) : (instr*env) → config → Prop :=
| sos_assign : ∀ l x e n ρ1 ρ2,
  sem_expr p ρ1 e n → subst ρ1 x n ρ2 → In x (vars p) →
  sos p (Assign l x e,ρ1) (Final ρ2)
[...]
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[...]
```

Reachable states from any initial environment

```
Inductive reachable_sos (p:program) : pp*env → Prop
  := [...]
```

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Each abstract element is given a concretization in $\mathcal{P}(\mathrm{pp} \times \mathrm{env})$

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Definition γ : abdom → (pp*env → Prop) := [...]
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Definition analyse : program → abdom := [...]
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Each abstract element is given a concretization in $\mathcal{P}(\mathrm{pp} \times \mathrm{env})$

```
Definition γ : abdom → (pp*env → Prop) := [...]
```

The analyzer must compute a correct over-approximation of the reachable states

```
Theorem analyse_correct : ∀ prog:program,
  reachable_sos prog ⊆ γ (analyse prog).
```

Roadmap

Standard Semantics

Abstract Semantics

Roadmap

Standard Semantics

direct soundness proof →

Abstract Semantics

Previous works:

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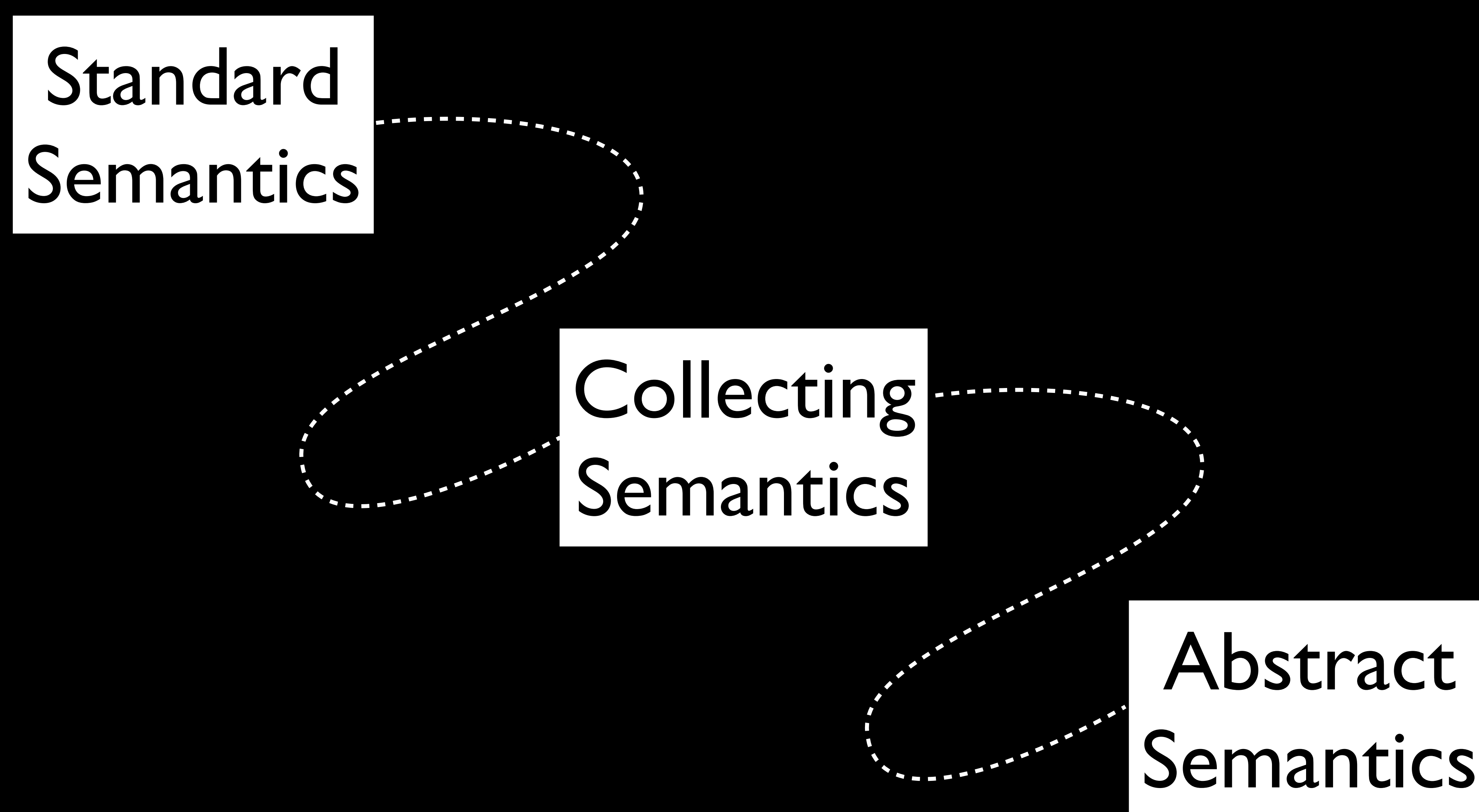
Roadmap

Standard Semantics

Collecting Semantics

Abstract Semantics

Roadmap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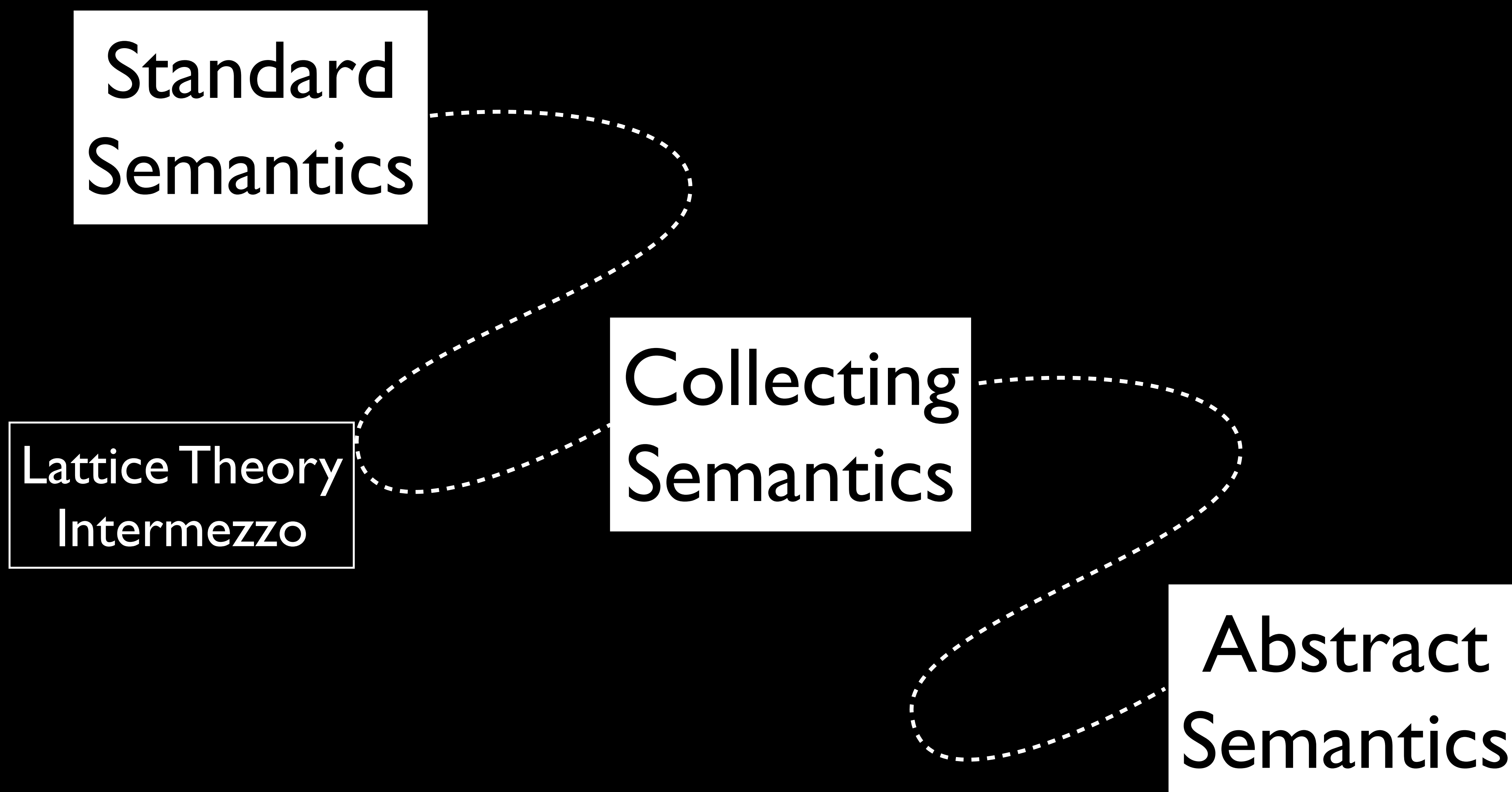
Roadmap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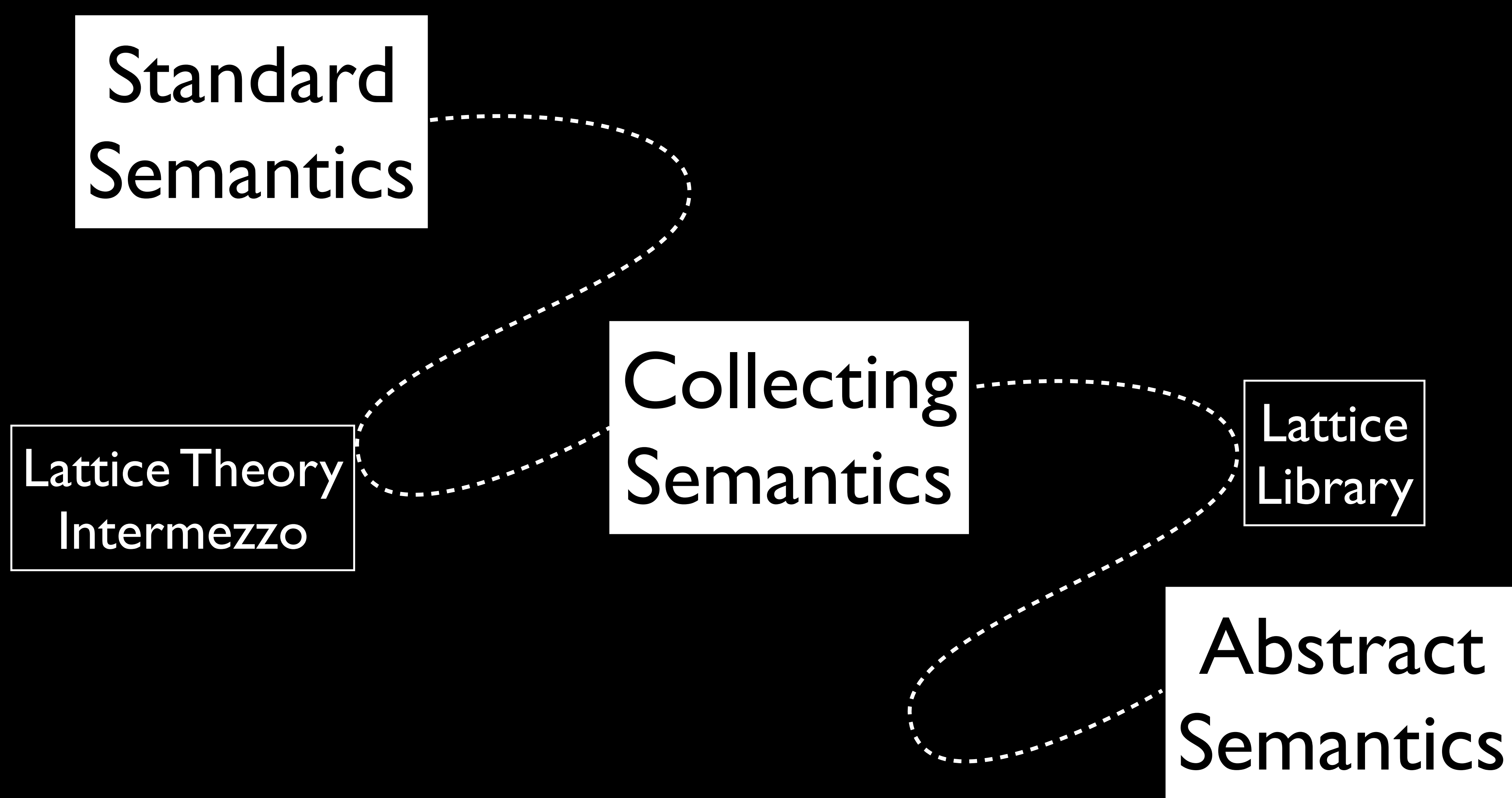

Standard Semantics
Lattice Theory Intermezzo
Collecting Semantics
Lattice Library
Abstract Semantics

Lattice Theory Intermezzo

A Few Lattice Theory

We need a least-fixpoint operator in Coq

- Formalization of complete lattices
- Proof of Knaster-Tarski theorem
- Construction of some useful complete lattices

Knaster-Tarski Theorem

```
Definition lfp `{CompleteLattice.t L} (f:monotone L L) : L :=
  CompleteLattice.meet (PostFix f).
```

Knaster-Tarski Theorem

Complete lattices on elements of type A

Monotone functions from L to L

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```

$$\bigsqcap \{x \mid f(x) \sqsubseteq x\}$$

Monotone functions

```
Class monotone A {Poset.t A} B {Poset.t B} : Type := Mono {
  mon_func : A → B;
  mon_prop : ∀ a1 a2,
    a1 ⊑ a2 → (mon_func a1) ⊑ (mon_func a2)
}.
```

A monotone function is a term
(Mono f π)

Knaster-Tarski Theorem

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Section KnasterTarski.
  Variable L : Type.
  Variable CL : CompleteLattice.t L.
  Variable f : monotone L L.
  Lemma lfp_fixpoint : f (lfp f) == lfp f. [...]
  Lemma lfp_least_fixpoint : ∀ x, f x == x → lfp f ⊑ x. [...]
  Lemma lfp_postfixpoint : f (lfp f) ⊑ lfp f. [...]
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End KnasterTarski.
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End KnasterTarski.
```

Coq Type Classes = Record + Inference (super) capabilities

Knaster-Tarski Theorem

We declare this argument as implicit

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Coq Type Classes = Record + Inference (super) capabilities

The implicit argument of type `(CompleteLattice.t L)` is automatically inferred

Canonical Complete Lattices

```
Instance PowerSetCL A : CompleteLattice.t 𝒫(A) := [...]


Instance PointwiseCL A L {CompleteLattice.t L} :
                        CompleteLattice.t (A → L) := [...]
```

Canonical Complete Lattices

Notation for (A->Prop)

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Functor

Pointwise ordering

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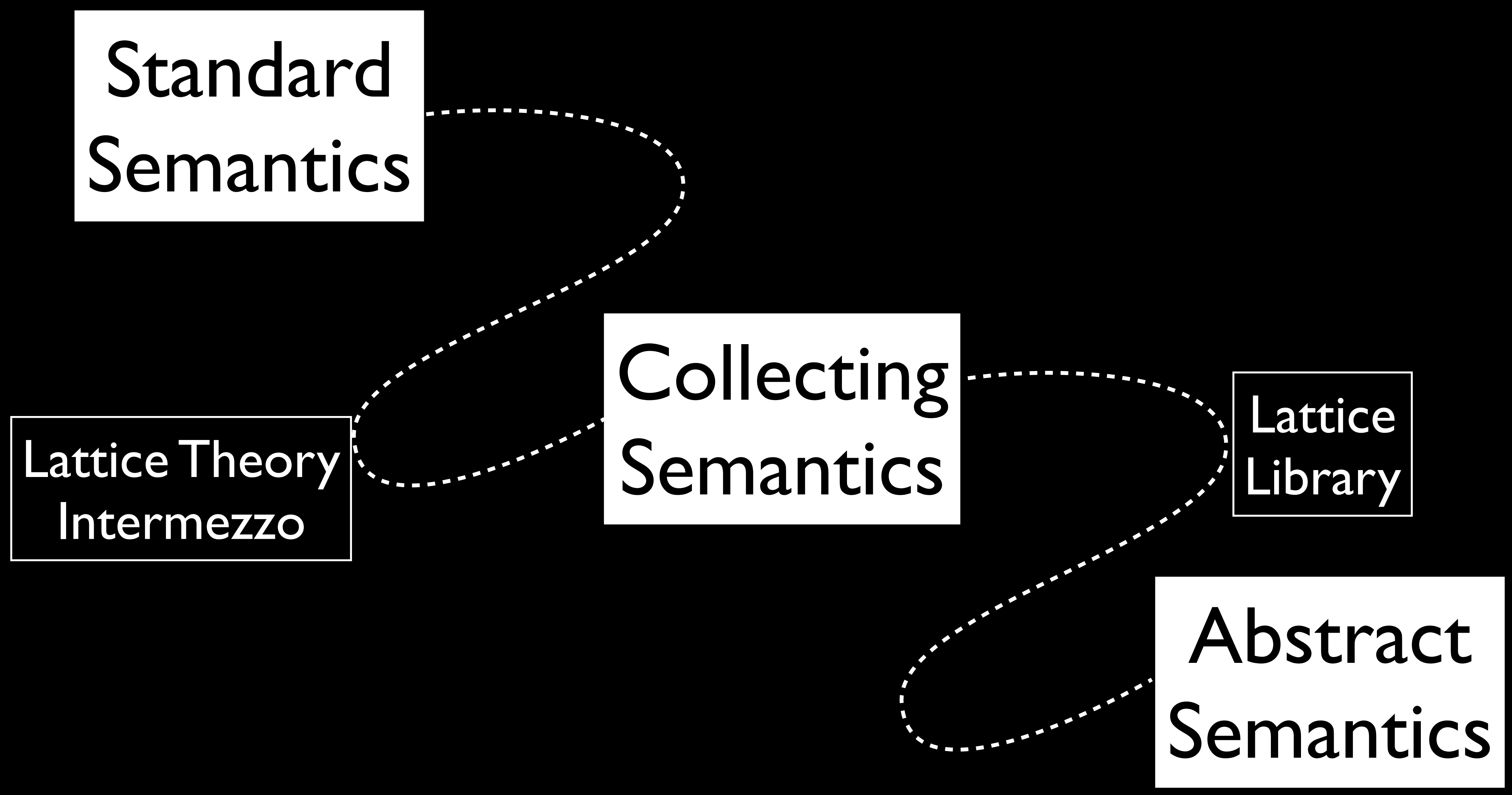
Definition example (f:monotone (B → 𝒫(C)) (B → 𝒫(C))) :=
  lfp f.
```

The right complete lattice is automatically inferred

Roadmap


Standard Semantics
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Abstract Semantics

Collecting Semantics

- An important component in the Abstract Interpretation framework
- Mimics the behavior of the static analysis (fixpoint iteration)
- But still in the concrete domain
- Similar to a denotational semantics but operates on $\wp(State)$ instead of $State_{\perp}$

Collecting Semantics: Example

```
i = 0; k = 0;

while  k < 10  {

   i = 0 ;

  while  i < 9  {

     i = i + 2
  };

   k = k + 1
}
```

Collecting Semantics: Example

```
i = 0; k = 0;

while [k < 10]^l1 {

   [i = 0]^l2;

   while [i < 9]^l3 {

      [i = i + 2]^l4
   };

   [k = k + 1]^l5
}_l6
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```

$$l_1 \mapsto [0, 10] \times ([0, 10] \cap \mathrm{Even})$$

$$l_2 \mapsto [0, 9] \times ([0, 10] \cap \mathrm{Even})$$

$$l_3 \mapsto [0, 9] \times ([0, 10] \cap \mathrm{Even})$$

$$l_4 \mapsto [0, 9] \times ([0, 8] \cap \mathrm{Even})$$

$$l_5 \mapsto [0, 9] \times ([0, 10] \cap \mathrm{Even})$$

$$l_6 \mapsto \{(10, 10)\}$$

Collecting Semantics

precondition

`Collect (i:stmt) (l:pp) :` $\mathcal{P}$`(env)` $\rightarrow$ `(pp` $\rightarrow \mathcal{P}$`(env))`

label after i

invariants on each reachable states during execution of i

Collecting Semantics

$$\texttt{Collect (i:stmt) (l:pp) : monotone (}\mathcal{P}\texttt{(env)) (pp} \rightarrow \mathcal{P}\texttt{(env))}$$

We generate only
monotone operators

Collecting Semantics

`Collect (i:stmt) (l:pp) : monotone` $(\mathcal{P}$`(env))` `(pp` $\rightarrow \mathcal{P}$`(env))`

Final instanciation:

`Collect p.(p_stmt) p.(p_end)` $\top$ `: (pp` $\rightarrow \mathcal{P}$`(env))`

invariants on each reachable states

Collecting Semantics

```
Program Fixpoint Collect (i:stmt) (l:pp):
                             monotone (𝒫(env)) (pp → 𝒫(env)) :=
match i with
  | Assign p x e =>
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  | While p t i =>
    [...]




  | [...]
end.
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map substitution + union

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map substitution + union

Strongest post-condition assignment transformer

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   | While p t i =>
     Mono (fun Env =>
          let I:𝒫(env) := lfp      ?                        in
            (Collect i p (assert t I))
                 +[p ↦ I] +[l ↦ assert (Not t) I]) _

   | [...]
end.
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Fixpoint equation: `I == Env ⊔ (Collect i p (assert t I) p)`

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must be monotone

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    Mono (fun Env => ⊥ +[p ↦ Env] +[l ↦ assign x e Env]) _

  | While p t i =>
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          let I:𝒫(env) := lfp (iter Env (Collect i p) t p) in
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  | [...]
end.
```

proof obligation

must be monotone

proof obligation

Collecting Semantics

```
Program Fixpoint Collect (i:stmt) (l:pp):
                          monotone (𝒫(env)) (pp → 𝒫(env)) :=
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   | Assign p x e =>
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   | [...]
end.
```

proof obligation

must be monotone

proof obligation

Proof obligations are generated by the `Program` mechanism and then automatically discharged by a custom tactic for monotonicity proofs

Collecting Semantics

```
Definition reachable_collect (p:program) (s:pp*env) : Prop :=
  let (k,env) := s in
    Collect p p.(p_instr) p.(p_end) (⊤) k env.

Theorem reachable_sos_implies_reachable_collect :
  ∀ p, reachable_sos p ⊆ reachable_collect p.
```

Collecting Semantics

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Definition reachable_collect (p:program) (s:pp*env) : Prop :=
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  ∀ p, reachable_sos p ⊆ reachable_collect p.
```
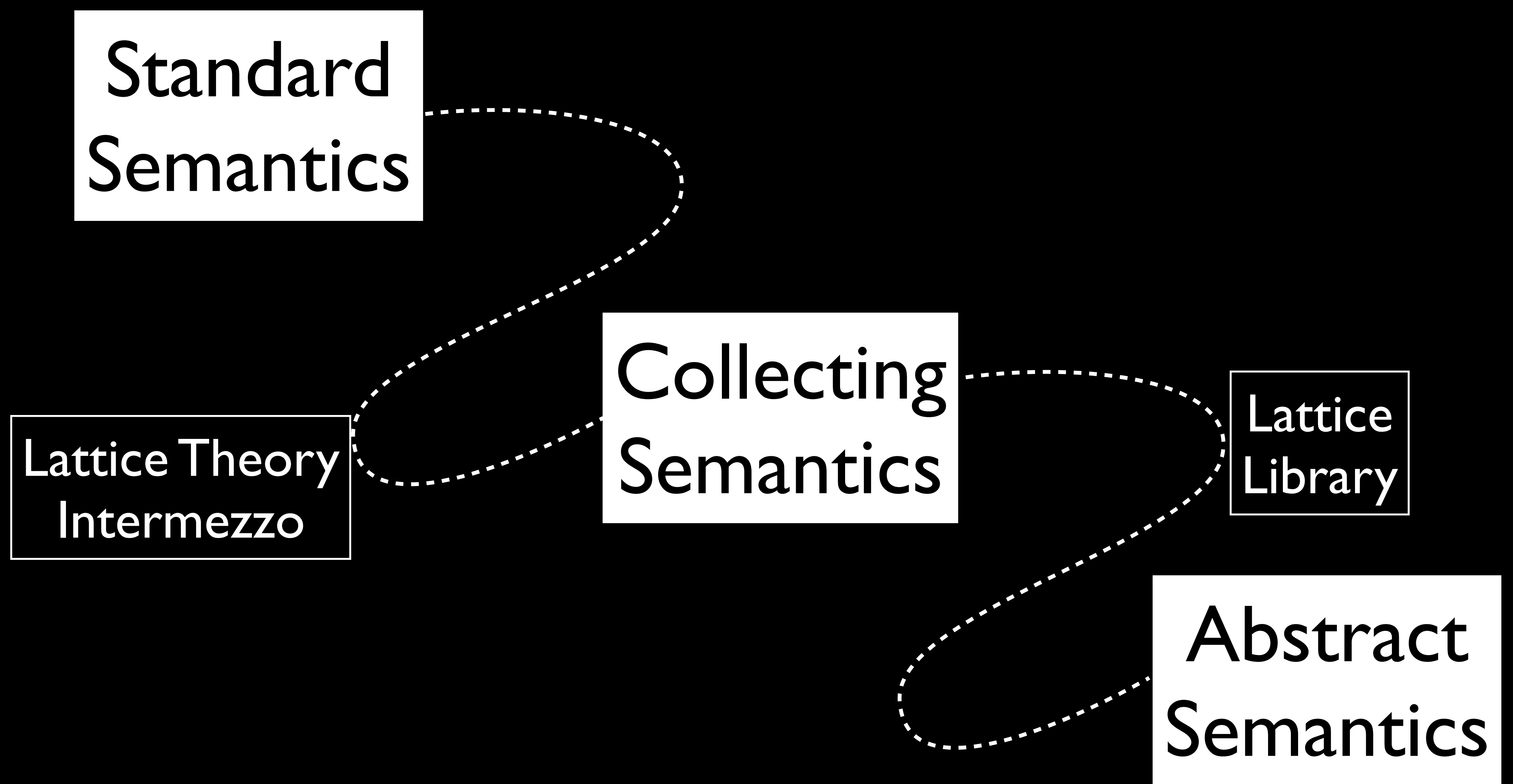

This is the most difficult proof of this work. It is sometimes just skipped in the AI literature because people *start* from a collecting semantics.

Roadmap


Standard Semantics
Lattice Theory Intermezzo
Collecting Semantics
Lattice Library
Abstract Semantics

Abstract Lattices

- Nothing can be extracted from the collecting semantics
  - it operates on Prop
  - that's why we were able to *program* the *not-so-constructive* lfp operator in Coq
- The abstract semantics will not computes on $\texttt{(pp} \to \mathcal{P}\texttt{(env))}$ but on an *abstract lattice* $\texttt{A}^\sharp$

Abstract Lattice

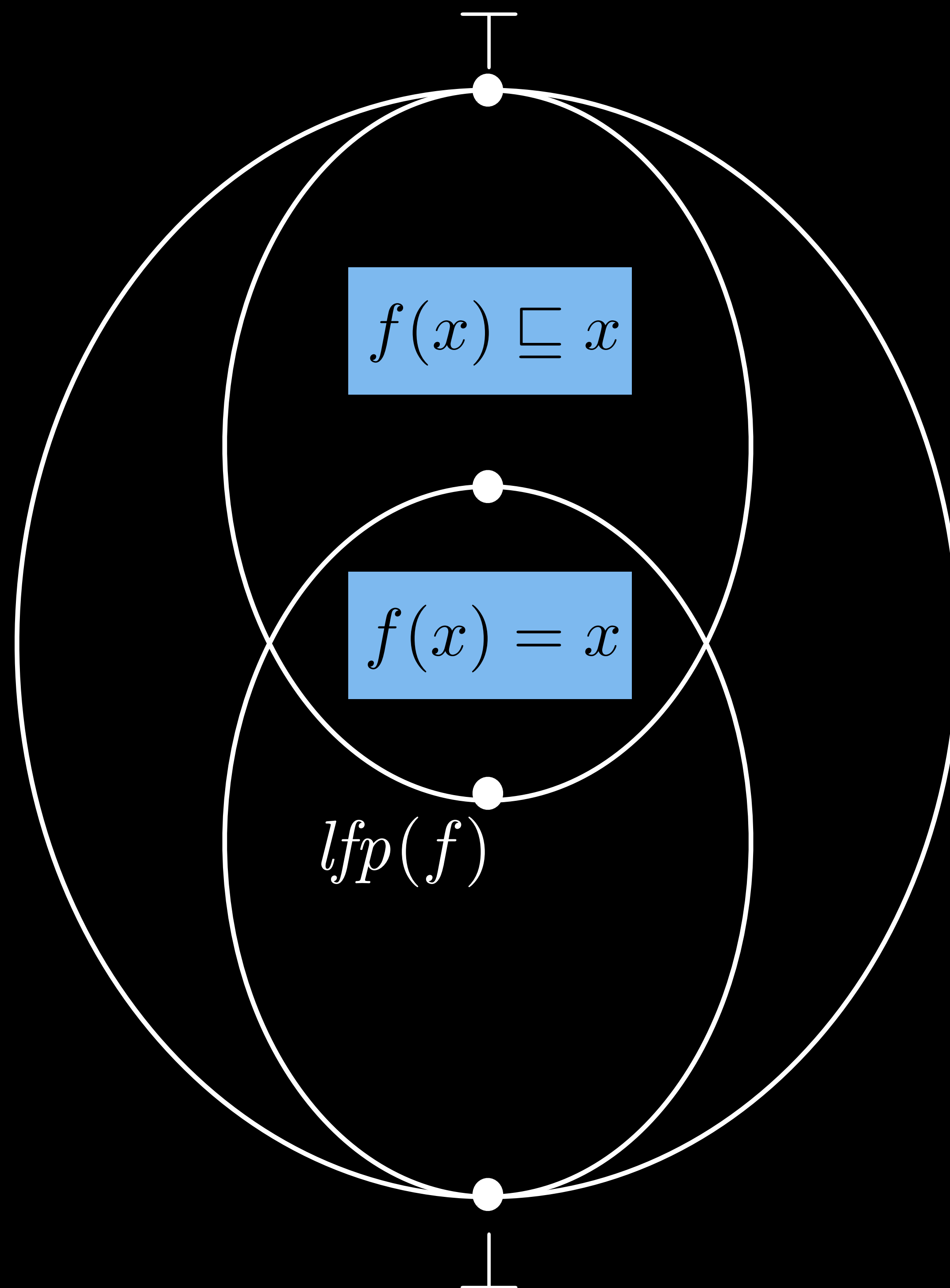
Abstract lattices are formalized with type classes

AbLattice.t : $\sqsubseteq^\sharp, \sqcap^\sharp, \sqcup^\sharp, \bot^\sharp$ + widening/narrowing

Each abstract lattice is equipped with a post-fixpoint solver

```
Definition approx_lfp :
    ∀ `{AbLattice.t t}, (t → t) → t := [...]

Lemma approx_lfp_is_postfixpoint :
    ∀ `{AbLattice.t t} (f:t → t),
         f (approx_lfp f) ⊑♯ (approx_lfp f).
```

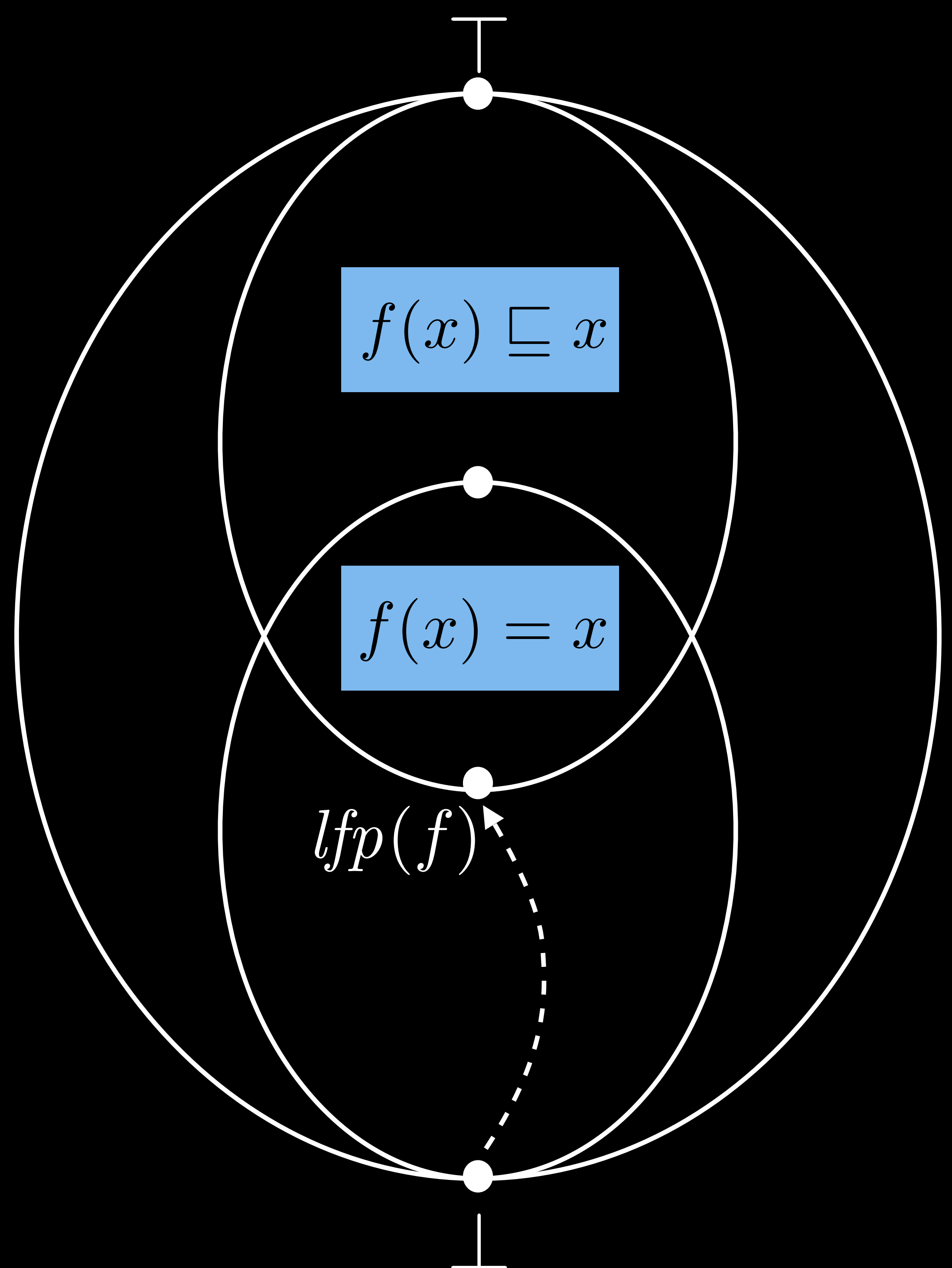
Fixpoint approximation with widening/narrowing



Fixpoint approximation with widening/narrowing

Standard Kleene fixed-point theorem

- too slow for big lattices (or just infinite)



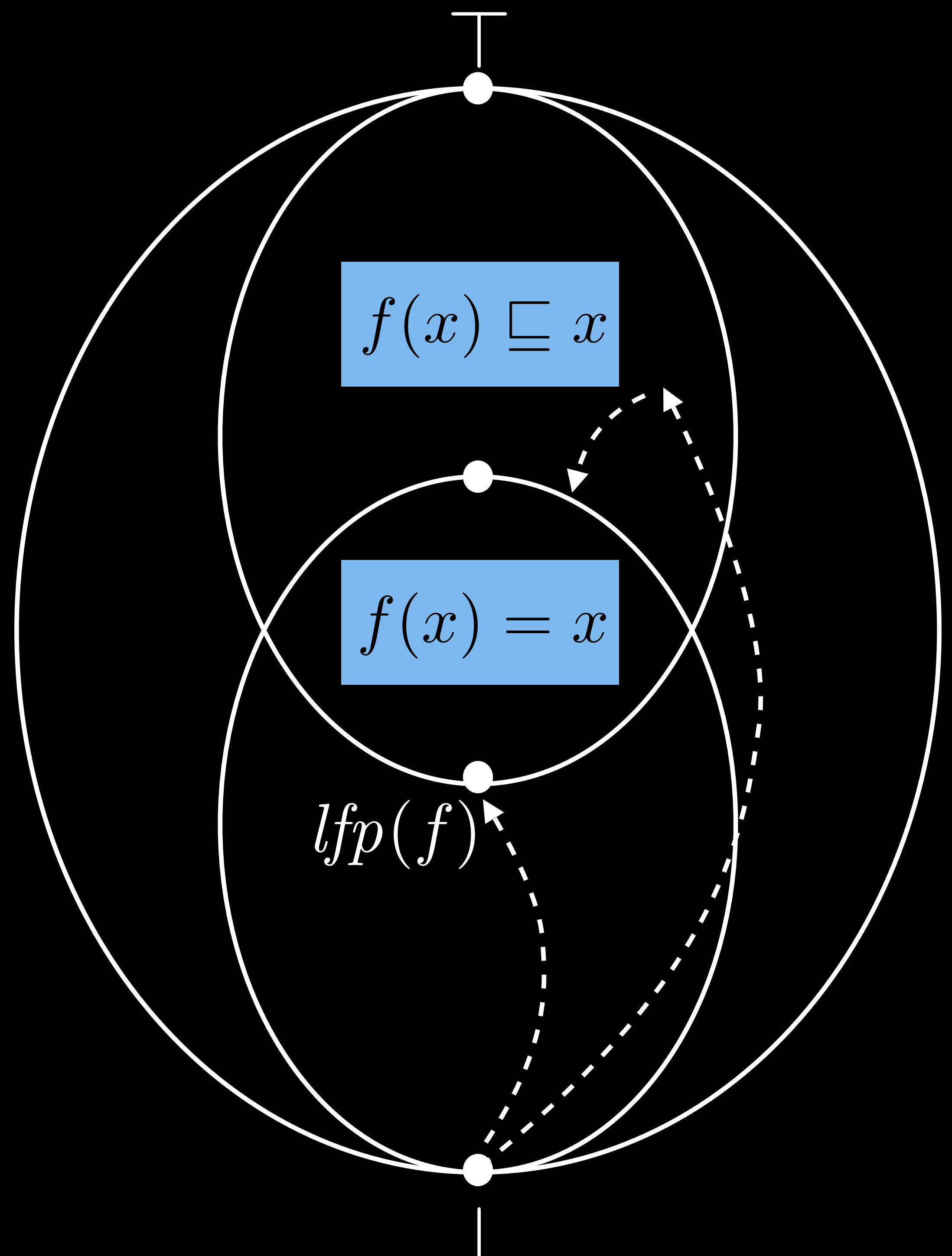
Fixpoint approximation with widening/narrowing

Standard Kleene fixed-point theorem

- too slow for big lattices (or just infinite)

Fixpoint approximation by widening/ narrowing

- over-approximates the lfp.
- requires different termination proofs than ascending chain condition
- on fixpoint equations, iteration order matters a lot !



Abstract Lattice

A library[1] is provided to build complex lattice objects with various functors for products, sums, lists and arrays.

Examples:

```
Instance ProdLattice
  t1 t2 {L1:AbLattice.t t1} {L2:AbLattice.t t2}:
  AbLattice.t (t1*t2) := [...]


Instance ArrayLattice t {L:AbLattice.t t}:
  AbLattice.t (array t) := [...]
```

---

[1] Adapted from our previous work: *Building certified static analysers by modular construction of well-founded lattices*. FICS'08

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Contains a difficult termination proof !

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Functional maps

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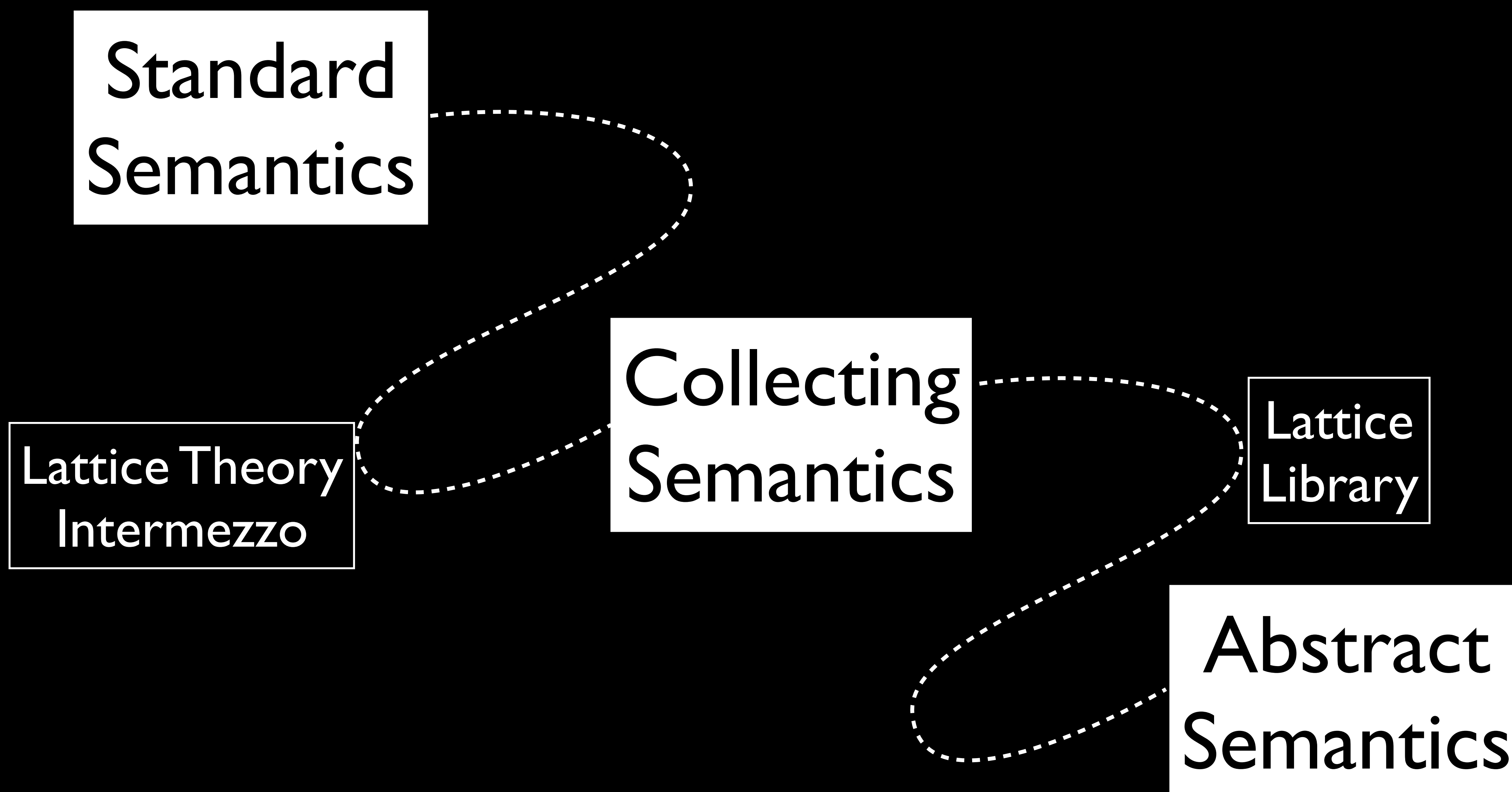
Roadmap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The analyzer is parameterized wrt. to an environment abstraction.

The development provides several non-relational instantiations.

Abstract Algebra

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The development provides several non-relational instantiations.

```
i = 0; k = 0;
          k ∈ [0,10]   i ∈ [0,10]
while k < 10 {
          k ∈ [0,9]   i ∈ [0,10]
  i = 0;
          k ∈ [0,9]   i ∈ [0,10]
  while i < 9 {
          k ∈ [0,9]   i ∈ [0,8]
    i = i + 2
  };
          k ∈ [0,9]   i ∈ [9,10]
  k = k + 1
}
          k ∈ [10,10]   i ∈ [0,10]
```

interval

Abstract Algebra

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  k = k + 1
}
              k ∈ [10,10]   i ∈ [0,10]
```

interval

```
i = 0; k = 0;
              k ≥ 0   i ≥ 0
while k < 10 {
              k ≥ 0   i ≥ 0
  i = 0;
              k ≥ 0   i ≥ 0
  while i < 9 {
              k ≥ 0   i ≥ 0
    i = i + 2
  };
              k ≥ 0   i > 0
  k = k + 1
}
              k > 0   i ≥ 0
```

sign

Abstract Algebra

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The development provides several non-relational instantiations.

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i = 0; k = 0;
```
$k \in [0, 10] \quad i \in [0, 10]$
```
while k < 10 {
```
$k \in [0, 9] \quad i \in [0, 10]$
```
  i = 0;
```
$k \in [0, 9] \quad i \in [0, 10]$
```
  while i < 9 {
```
$k \in [0, 9] \quad i \in [0, 8]$
```
    i = i + 2
  };
```
$k \in [0, 9] \quad i \in [9, 10]$
```
  k = k + 1
}
```
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interval

```
i = 0; k = 0;
```
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```
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```
$k \geq 0 \quad i \geq 0$
```
  while i < 9 {
```
$k \geq 0 \quad i \geq 0$
```
    i = i + 2
  };
```
$k \geq 0 \quad i > 0$
```
  k = k + 1
}
```
$k > 0 \quad i \geq 0$

sign

```
i = 0; k = 0;
```
$i \equiv 0 \mod 2$
```
while k < 10 {
```
$i \equiv 0 \mod 2$
```
  i = 0;
```
$i \equiv 0 \mod 2$
```
  while i < 9 {
```
$i \equiv 0 \mod 2$
```
    i = i + 2
  };
```
$i \equiv 0 \mod 2$
```
  k = k + 1
}
```
$i \equiv 0 \mod 2$

congruence

Abstract Semantics

```
Section prog.
Variable (t : Type) (L : AbLattice.t t)
         (prog : program) (Ab : AbEnv.t L prog).

Fixpoint AbSem (i:instr) (l:pp) : t → array t :=
match i with
| Assign p x e =>
  fun Env => ⊥♯ +[p ↦ Env]♯ +[l ↦ Ab.assign Env x e]♯
| While p t i => fun Env =>
  let I := approx_lfp
            (fun X => Env ⊔♯
                      (get (AbSem i p (Ab.assert t X)) p)) in
  (AbSem i p (Ab.assert t I))
    +[p ↦ I]♯+[l ↦ Ab.assert (Not t) I]♯
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end.
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Abstract counterpart of
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| [...]
end.
```

Abstract counterpart of concrete operations

Fixpoint approximation instead of least fixpoint computation

Connecting Concrete and Abstract Semantics

```
Theorem AbSem_correct : ∀ i l_end Env,
  Collect prog i l_end (γ Env) ⊑ γ (AbSem i l_end Env).
```

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Soundness proof between abstract and collecting semantics

Type Classes to the rescue

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Theorem AbSem_correct : ∀ i l_end Env,
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```

Need 4 minutes
after

Type Classes to the rescue

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canonical order on pp → $\mathcal{P}$(env)

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Type Classes to the rescue

concretization on pp $\to \mathcal{P}$(env)

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concretization on `pp` $\to \mathcal{P}$`(env)`

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concretization on $\mathcal{P}$`(env)`

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```

concretization on $\mathcal{P}(\texttt{env})$

canonical order on $\texttt{pp} \rightarrow \mathcal{P}(\texttt{env})$

Without Type Classes

Need 4 minutes after

```
Theorem AbSem_correct : ∀ i l_end Env,
  (PointwisePoset (PowerSetPoset env)).(Poset.o
  (Collect prog i l_end (AbEnv.(AbEnv.gamma) Env))
  (FuncLattice.Gamma AbEnv.(AbEnv.gamma) (AbSem i l_end Env)).
```

Type Classes to the rescue

concretization on **pp** $\rightarrow \mathcal{P}$**(env)**

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concretization on $\mathcal{P}$**(env)**

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concretization on pp $\rightarrow \mathcal{P}$(env)

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The proof is easy because the two semantics are very similar

Abstract Semantics

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Program Fixpoint Collect (i:stmt) (l:pp): monotone (𝒫(env)) (pp → 𝒫(env)) :=
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    [...]
  end.
```

First Coq instance of the slogan

My abstract interpreter is correct by construction

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Fixpoint AbSem (i:instr) (l:pp) : t → array t :=
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  [...]
  end.
```

Final Theorem

```
Definition analyse : array t :=
  AbSem prog.(p_instr) prog.(p_end) (Ab.top).

Theorem analyse_correct : ∀ k env,
  reachable_sos prog (k,env) → γ (get analyse k) env.
```

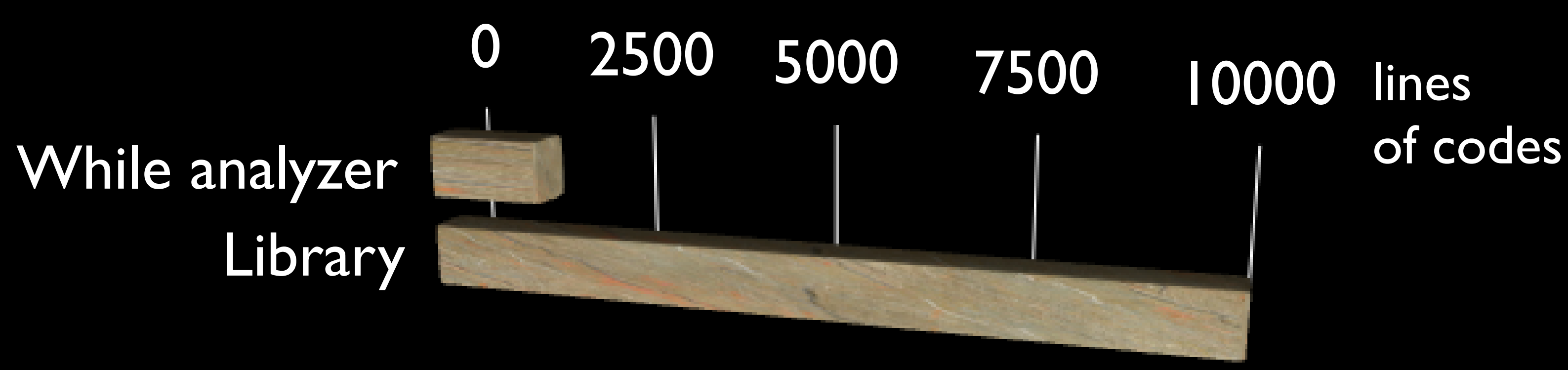
The function `analyse` can be extracted to real OCaml code

You can type-check, extract and run the analyser yourself !
http://www.irisa.fr/celtique/pichardie/ext/itp10/

Conclusions

The first mechanized proof of an abstract interpreter based on a collecting semantics

- requires lattice theory components
- provides a reusable library



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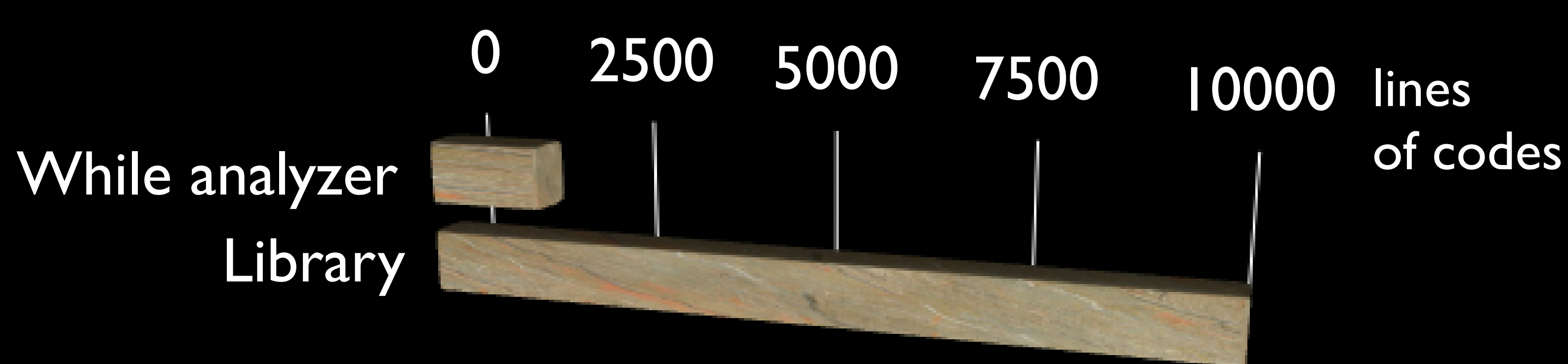

- the proof is more methodic and elegant than previous attempts

Well, of course this is matter of taste...

Perspectives

A first (small) step towards a certified Astrée-*like* analyser

- Ongoing project: scaling such an analyser to a C language
  - on top of the Compcert semantics
  - for a restricted C (no recursion, restricted use of pointers)

Abstraction Interpretation methodology

- would be nice to use more deeply the Galois connexion framework
- we prove soundness and termination: what about precision ?

Thanks !