Logical Foundations for the
ACL2 Theorem Prover

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Dept. of Computer Science

Joint work with Bob Boyer, J Moore,
and the ACL2 community

Presented at JAF 2019
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**MY ANSWERS:**

1. Introduce ACL2 as a **practical application** of logic.
2. Discuss **foundational issues** for ACL2.
OUTLINE

Overview and Context

Introduction to the ACL2 System

Logical Foundations for ACL2

Conclusion
Overview and Context

The ACL2 home page begins with the following summary.

ACL2 is a logic and programming language in which you can model computer systems, together with a tool to help you prove properties of those models. “ACL2” denotes “A Computational Logic for Applicative Common Lisp”.

5/41
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But before we talk about ACL2, let’s put it in context.
**Formal Verification**

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FV tools include *equivalence checkers, model checkers, various static checkers*, and (occasionally) *interactive theorem provers* (ITPs) such as Coq, Isabelle, HOL4, PVS, Agda — and **ACL2**.
INTERACTIVE THEOREM PROVING

- Yearly ITP conference
- ITP is typically more scalable than fully automatic tools, but it requires human assistance.
- In ACL2, one proves lemmas that may be used automatically to simplify terms in later proofs.
- Some strengths of ACL2 among ITPs:
  - Proof automation and debugging
  - Fast execution of programs
  - Documentation in hypertext format (120,000 lines for system; many more for libraries)
  - Scalability (see next slide)
Interactive Theorem Proving

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ACL2 has been used not only at universities and the U.S. Government, but also at several companies [4]:

- AMD, ARM, ArterisIP, Battelle, Centaur, GE, IBM, Intel, NXP, Kestrel, Oracle, Rockwell Collins

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"Microprocessor design goes daily through numerous optimizations that affect thousands of lines of code. These optimizations must be proved correct."

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A recent example of an ACL2 formalization at UT Austin:

- An efficient checker for Boolean satisfiability (SAT) proofs
- Used in recent international SAT competitions
- Has checked 2-petabyte SAT proof of longstanding open problem (Schur number 5) [3]; ∼16 CPU years
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  - The ACL2 community contributes with feature requests and (on occasion) prototype implementations.
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  [DEMO]: file demo-1.lsp
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  [DEMO]: file *demo-1.lsp*
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  [DEMO]: file *demo-2.lsp*
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- **ACL2 as an automated theorem prover**
  [DEMO]: file *demo-2.lsp*
  (log *demo-2-log.txt*)
  - **ACL2 provides automation** for induction, linear arithmetic, Boolean reasoning, rule application, . . .
  - During a proof, each goal is replaced by a list of subgoals (possible empty) such that if they are all theorems, then that goal is a theorem.
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(Anyhow, it’s nice to have Ken Kunen’s Nqthm proof of the Paris-Harrington theorem. [9])
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- numbers (complex rationals);
- characters;
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- symbols; and
- closure under an ordered pair operation, `cons`. 

ACL2 `(cons 3 nil)`
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Cons provides lists, with the symbol `nil` for the empty list.

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- We will see the importance of introducing new concepts locally: justified by conservativity.
- Theories evolve by introducing new function symbols using the extension principles. [6]
EXTENSION PRINCIPLE: DEFINITIONS

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A definition may be recursive if some *measure* into $\varepsilon_0$ is proved to decrease on each recursive call.
EXTENSION PRINCIPLE: CHOICE (AND ∃)

Quantification is implemented using a choice operator. When asked to define

\[ P(\vec{x}) = \exists \vec{y} A(\vec{x}, \vec{y}) \]

then ACL2 generates the following.
**Extension Principle: Choice (and ∃)**

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**Conservatively introduce** a Skolem (witness) function \( w(\vec{x}) \) and a predicate \( P(\vec{x}) \):

\[
\begin{align*}
  w(\vec{x}) &= \varepsilon \vec{y} A(\vec{x}, \vec{y}) \quad \text{[If any } \vec{y} \text{ satisfies } A(\vec{x}, \vec{y}), \text{ then } w(\vec{x}) \text{ does.]} \\
  P(\vec{x}) &= A(\vec{x}, w(\vec{x}))
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\[ P(\vec{x}) = A(\vec{x}, w(\vec{x})) \]

(defun-sk fermat-counterex (n)
  (exists (i j k)
    (and (posp i) (posp j) (posp k)
      (equal (+ (expt i n) (expt j n))
        (expt k n))))
(deffthm fermat
  (implies (and (integerp n) (< 2 n))
    (not (fermat-counterex n))))
EXTENSION PRINCIPLE: CHOICE (AND ∃) (2)

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Extension principle: Choice (and \( \exists \)) (2)

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Conservativity \textit{with} induction follows from a model-theoretic forcing argument.
**Extension principle: Constraints**

It is also legal to introduce *constrained* functions, using axioms that are *proved* about *local witnesses.*
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**Example:**

```lisp
(encapsulate ((fn (x y) t))
    (local (defun fn (x y)
        (+ x y)))
    (defthm fn-commutative
        (equal (fn x y) (fn y x))))
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A derived inference rule, functional instantiation [2], is often useful with constrained functions.

Example:
(defun map2-fn (lst1 lst2)
  (if (consp lst1)
      (cons (fn (first lst1) (first lst2))
            (map2-fn (rest lst1) (rest lst2)))
      nil))

(deffthm map2-fn-commutative
  (implies (equal (len lst1) (len lst2)); same length
            (equal (map2-fn lst2 lst1)
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(defun map2-* (lst1 lst2)
    (if (consp lst1)
        (cons (* (first lst1) (first lst2))
              (map2-* (rest lst1) (rest lst2)))
        nil))

(defthm map2-*--commutative
    (implies (equal (len lst1) (len lst2))
              (equal (map2-* lst2 lst1)
                    (map2-* lst1 lst2)))
    :hints ("Goal" :by (:functional-instance
                         map2-fn-commutative
                         (fn *) (map2-fn map2-*)())))
**CONSERVATIVITY AND LOCAL**

Fun example in **ACL2(r)**, a variant of ACL2 that supports the real numbers, due to Ruben Gamboa:
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Informally:
If internal predicate $P(n, x)$ holds for all standard natural numbers $n$, then $P(n, x)$ holds for some non-standard natural number $n$. 
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Using **LOCAL** can dramatically speed up book inclusion!
(local ; Hence skipped when including this top-level book!
  (include-book "overspill-proof"))

(defstub overspill-p (n x) t)

(defun overspill-p* (n x)
  (if (zp n)
      (overspill-p 0 x)
      (and (overspill-p n x)
           (overspill-p* (1- n) x))))

(defchoose overspill-p-witness (n) (x)
  (or (and (natp n) (standardp n)
           (not (overspill-p n x)))
      (and (natp n) (i-large n)
           (overspill-p* n x))))

(defthm overspill-p-overspill
  (let ((n (overspill-p-witness x)))
    (or (and (natp n) (standardp n)
             (not (overspill-p n x)))
        (and (natp n) (i-large n)
             (implies (and (natp m)
                            (<= m n))
                      (overspill-p m x))))))

:rule-classes nil)
Meta-theoretic Reasoning (1)

In ACL2, you can [1, 5]:

Efficient execution can be important for meta-theoretic reasoning!
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ACL2 !>(loop$ for i in '(3 5 7) sum (* i i))
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ACL2 gives the following semantics to the second of these.

(sum$ '(lambda (i) (* i i))
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```
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 '(3 5 7))
```

where **sum$** is defined essentially as follows.

```
(defun sum$ (fn lst)
  (if (endp lst) ; lst is empty
      0
      (+ (apply$ fn (list (first lst)))
         (sum$ fn (rest lst)))))
```
"HIGHER-ORDER" Apply$ (1)

We cannot employ the usual two-sorted, weak second-order approach. **Example**: Not a theorem without the defun!

```lisp
(local (defun f (x) x))
(thm (equal (apply$ 'f (list x)) x))
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```

**Example successful use of** `apply$`:

```lisp
(include-book "projects/apply/top" :dir :system)
(defun$ norm^2 (x y) (+ (* x x) (* y y)))
(assert-event (equal (norm^2 3 4) 25))
(thm (equal (norm^2 3 4) 25))
(assert-event (equal (apply$ 'norm^2 (list 3 4)) 25))
```
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Example successful use of apply$:

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But the following fails, as it should:
apply$ is a constrained function with trivial constraints.

(thm (equal (apply$ 'norm^2 (list 3 4)) 25))
“HIGHER-ORDER” \texttt{Apply$ (2)}

However, the proof succeeds for the \texttt{thm} below, where the \textit{warrant hypothesis}, \texttt{(warrant norm^2)}, \textbf{asserts}:
\[
\forall x\ y \ (\text{equal} \ (\text{apply$ 'norm^2 \ (\text{list} \ x \ y))} \\
\text{ (norm^2} \ x \ y)).
\]

\[
\texttt{(thm} \ (\text{implies} \ (\text{warrant} \ \text{norm}^2) \ \\
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\text{ 25))))
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However, the proof succeeds for the thm below, where the warrant hypothesis, (warrant norm^2), asserts:
\[(\forall x y) (equal (apply$ 'norm^2 (list x y)) (norm^2 x y)).\]

(thm (implies (warrant norm^2)
  (equal (apply$ 'norm^2 (list 3 4)) 25))

Warrant hypotheses are not vacuous!
There is a natural evaluation theory where every warrant is attached to the constant “true” function. [8]
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- **Constraint for** a “specification” function, `spec`:
  \[ x \in \mathbb{Z} \implies spec(x) \in \mathbb{Z} \]
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- **Attach** \( \text{impl} \) to \( \text{spec} \): \( \text{defattach spec impl} \)

**Meaning:** \( (\forall x)(\text{spec}(x) = \text{impl}(x)) \)
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```
ACL2 :> (f 3 4) ; = spec(7) = impl(7)
70
ACL2 :>
```
DEFATTACH (2)

Issues to consider:

- Is \((\text{local defattach ...})\) supported?
  - \textbf{YES,} local is supported.
- Then how do we deal with conservativity?
  - Two theories: The current theory for reasoning and a stronger evaluation theory, extended using defattach:
    \[
    \forall x (\text{spec } x = \text{impl } x)
    \]
- Ah, but what about this?
  - \((\text{thm (equal (f 3 4) 70)})\)
    - The proof fails! (Good!)
- Is the evaluation theory consistent?
  - \textbf{Yes,} where the attachment relation must be acyclic.
  - Details: see Essay on Defattach comment in the ACL2 sources.
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Details: see *Essay on Defattach* comment in the ACL2 sources.
Some More Logical Challenges

Practical considerations create some more logical challenges.
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- **Packages** are a programming convenience but introduce axioms such as the following: *not conservative!*
  
  \[
  \text{symbol-package-name('PKG1::F) = "PKG1"}
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  Hence packages **must be recorded.**
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- One can specify a *measure* in order to admit a recursive definition. But what if the measure is defined in terms of a function whose definition is *local*?

- *Congruence-based reasoning* allows replacing one subterm by another that is equivalent but not necessarily equal. [7]
OUTLINE

Overview and Context

Introduction to the ACL2 System

Logical Foundations for ACL2

Conclusion
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- As an ITP system, it relies on user guidance for large problems but enjoys scalability.
- Logic provides critical foundational support for practical theorem proving software.
- For more information, see the ACL2 home page, in particular links to The Tours and Publications, which links to introductory material.


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Slides for this talk are available via links from my home page:
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Slides for this talk are available via links from my home page:
http://www.cs.utexas.edu/users/kaufmann

THANK YOU!
EXTRA SLIDES

We can go on, time permitting....
Some ACL2 features *not* discussed further today:

- **Prover algorithms**
  - Waterfall, linear arithmetic, Boolean reasoning, ...
  - Rewriting: Conditional, congruence-based, rewrite cache, syntaxp, bind-free, ...

- **Using the prover effectively**
  - The-method and introduction-to-the-theorem-prover
  - Theories, hints, rule-classes, ...
  - Accumulated-persistence, brr, proof-checker, dmr, ...

- **Programming support, including (just a few):**
  - Guards
  - Hash-cons and function memoization
  - Packages
  - Mutable State, stobjs, arrays, applicative hash tables, ...

- **System-level:** Emacs support, books and certification, abbreviated printing, parallelism (ACL2(p)), ...
META-THEORETIC REASONING (2)

ACL2 supports a notion of “evaluation”, together with this sort of *meta* theorem, directing the use of `fn` to transform terms that are calls of `nth` or `of foo`.
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\begin{verbatim}
(defthm fn-correct-1
  (equal (evl x a)
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ON EFFICIENT EXECUTION

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- ACL2 definitions are actually programs in the Common Lisp programming language.

- Guards specify intended domains of functions and support sound, efficient Common Lisp evaluation.

- Several features support efficient computation by reusing storage, yet with a first-order logic foundation.
  - Single-threaded objects including state
  - Arrays
  - Function memoization (reuse of saved results)
  - Fast alists (applicative hash tables)