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An Extension of Moore's Result for Closed Queuing Networks

Abstract: In this communication, Moore's result for the normalization constant of a closed queuing network of exponential servers is extended to the case of nondistinct traffic intensities. We then show that this result can be applied to various more general closed and semiclosed queuing networks with a product-form solution.

Introduction

Substantial progress has been made recently in extending the scope of Jackson's product-form solution for queuing networks [1-5]. At the same time, a number of very efficient computational algorithms [3, 6-8] have been developed for evaluating the normalization constant and marginal queue length statistics for such networks. These algorithms are all based upon some recursive scheme. For the special case of a closed network of exponential servers with distinct traffic intensities, Moore has shown an explicit solution for the normalization constant using the partial fraction method [9]. In this note, we extend Moore's result to the case of nondistinct traffic intensities and show that it can be applied to various more general closed and semiclosed queuing networks with the product-form solution. We note, however, that the partial fraction method is less versatile than recursive techniques. It also requires the summation of terms with alternating signs, and thus may be subject to round-off errors in some cases [3].

Assumptions

We outline here the scope of queuing networks for which the results in this communication are applicable. Consider a queuing network with M_s service stations and R classes of customers. (See [2].) At the completion of a service request, a customer may change its class membership and proceed to another service station (or leave the network) according to fixed transition probabilities. Four types of service stations may be considered: 1) a single server, first-come-first-served service discipline (FCFS); 2) a single server, processor-sharing service discipline (PS); 3) a single server, last-come-first-served preemptive-resume service discipline (LCFS); and 4) no queuing, arbitrarily many servers (IS). In an FCFS service station, all customers have the same negative exponential service time distribution with a fixed service rate. In PS, LCFS, or IS service stations, each class of customers may have its own general service time distribution which has a rational Laplace transform.

Let n_i, $1 \leq i \leq M_s$, denote the number of customers at service station i. We assume that the underlying Markov chain of customer transitions is irreducible. The arrival process of customers to the network is a Poisson process with an arrival rate dependent on the instantaneous number n^t of customers in the network at time t. Moreover, if $n^t = N_1$, a departing customer is immediately replaced; if $n^t = N_2$, all arrivals are lost. Thus, n^t is constrained: $N_1 \leq n^t \leq N_2$. In particular, for a closed queuing network $N_1 = N_2$. Below, we first present our results for closed queuing networks and then indicate how to extend these results to the more general case of semiclosed networks $(N_1 < N_2)$ [7].

Results for closed queuing networks

Let there be M service stations of types FCFS, PS, and LCFS labeled by $\{1, 2, \cdots, M\}$, and $M_s - M$ service stations of type IS labeled by $\{M+1, \cdots, M_s\}$. Suppose ρ_{ir}, $1 \leq i \leq M_s$ and $1 \leq r \leq R$, are the relative traffic intensities [3] of class r customers at service station i. Define

$$\rho_i \triangleq \sum_{r=1}^{R} \rho_{ir}, \qquad 1 \leq i \leq M_s.$$

The equilibrium joint queue length distribution is

$$P(n_1, \cdots, n_{M_s}) = C\left[\prod_{i=1}^{M} (\rho_i)^{n_i}\right]\left[\prod_{i=M+1}^{M_s} \frac{(\rho_i)^{n_i}}{n_i!}\right], \tag{1}$$

where C is a normalization constant chosen to make the equilibrium state probabilities sum to one. To solve for C, we first define

$$n_0 \triangleq \sum_{i=M+1}^{M_s} n_i$$

and

$$\rho_0 \triangleq \sum_{i=M+1}^{M_s} \rho_i.$$

From Eq. (1) we obtain the joint probability distribution

$$P(n_0, n_1, \cdots, n_M) = C \frac{(\rho_0)^{n_0}}{n_0!} \prod_{i=1}^{M} (\rho_i)^{n_i}.$$

Define

$$G(n) \triangleq \begin{cases} 1, & n = 0; \\ \displaystyle\sum_{\mathbf{n}\in\mathscr{S}(n)} \frac{(\rho_0)^{n_0}}{n_0!} \prod_{i=1}^{M} (\rho_i)^{n_i}, & n \geq 1, \end{cases} \tag{2}$$

where

$$\mathbf{n} \triangleq (n_0, n_1, \cdots, n_M),$$

and

$$\mathscr{S}(n) \triangleq \{\mathbf{n} \mid \sum_{i=0}^{M} n_i = n, \quad n_i \geq 0 \text{ for all } i\}.$$

For a closed network of N customers, $P(\mathbf{n})$, $\mathbf{n} \in \mathscr{S}(N)$ must sum to one. Thus the normalization constant is $C = [G(N)]^{-1}$.

• *Solution for $G(N)$*

We first define

$$g(n) \triangleq \begin{cases} 1, & n = 0; \\ \displaystyle\sum_{\mathbf{n}'\in\mathscr{S}'(n)} \prod_{i=1}^{M} (\rho_i)^{n_i}, & n \geq 1, \end{cases} \tag{3}$$

where

$$\mathbf{n}' \triangleq (n_1, n_2, \cdots, n_M),$$

$$\mathscr{S}'(n) \triangleq \{\mathbf{n}' \mid \sum_{i=1}^{M} n_i = n, \quad n_i \geq 0 \text{ for all } i\}.$$

For a queuing network with no IS service station, the normalization constant C is equal to $[g(N)]^{-1}$. Moore has shown an explicit solution for $g(n)$ under the assumption that the traffic intensities ρ_i are distinct [9]. In the Appendix, we extend Moore's derivation to the case of nondistinct traffic intensities. The results are as follows.

Let there be M' distinct ρ_i with multiplicity $m(i)$ such that

$$\sum_{i=1}^{M'} m(i) = M.$$

We have then

$$g(n) = \sum_{i=1}^{M'} (\rho_i)^n \sum_{j=0}^{m(i)-1} \binom{n + m(i) - j - 1}{n} A_{ij}, \tag{4}$$

where

$$A_{i0} = \prod_{k=1,k\neq i}^{M'} \{[1 - (\rho_k/\rho_i)]^{m(k)}\}^{-1}, \qquad i = 1, 2, \cdots, M', \tag{5}$$

and

$$A_{ij} = \frac{1}{j} \sum_{l=0}^{j-1} A_{il} (-1)^{j-l} \sum_{k=1,k\neq i}^{M'} m(k) \left[\frac{\rho_k}{\rho_i - \rho_k}\right]^{j-l}, \quad i = 1, 2, \cdots, M'; \quad j = 1, 2, \cdots, m(i) - 1. \tag{6}$$

Note that $M' \geq 2$ is assumed above. If $M' = 1$, then we simply have

$$g(n) = \binom{n + m(i) - 1}{n} \rho_1^n.$$

Example 1 If all ρ_i are distinct, the above solution for $g(n)$ reduces to Moore's result:

$$g(n) = \sum_{i=1}^{M} A_i \rho_i^n,$$

where

$$A_i = \left\{ \prod_{k=1,\, k\neq i}^{M} [1 - (\rho_k/\rho_i)] \right\}^{-1}.$$

Example 2 In a network with $M + 1$ service stations, $\rho_M = \rho_{M+1}$, and distinct ρ_i for $1 \leq i \leq M - 1$,

$$g(n) = \sum_{i=1}^{M-1} \frac{(\rho_i)^{n+M}}{(\rho_i - \rho_M) \displaystyle\prod_{k=1,k\neq i}^{M} (\rho_i - \rho_k)} + \frac{(\rho_M)^{n+M-1}}{\displaystyle\prod_{k=1}^{M-1} (\rho_M - \rho_k)} \left[n + 1 - \sum_{k=1}^{M-1} \frac{\rho_k}{\rho_M - \rho_k} \right].$$

To obtain $G(N)$, Eq. (2) may be rewritten as

$$G(N) = \sum_{n=0}^{N} g(n) \frac{(\rho_0)^{N-n}}{(N-n)!}.$$

By substituting Eq. (4) into the above and exchanging summations, we obtain

$$G(N) = \sum_{i=1}^{M'} (\rho_i)^N \sum_{n=0}^{N} \left[\frac{(\rho_0/\rho_i)^{N-n}}{(N-n)!} \times \sum_{j=0}^{m(i)-1} A_{ij} \binom{n + m(i) - 1 - j}{n} \right]. \tag{7}$$

Example 3 All ρ_i are distinct.

$$G(N) = \sum_{i=1}^{M} A_i (\rho_i)^N \sum_{n=0}^{N} \frac{(\rho_0/\rho_i)^n}{n!}.$$

- *Marginal queue length statistics for an FCFS, PS, or LCFS service station*

The marginal queue length probability distribution for an FCFS, PS, or LCFS service station is

$$P[n_i \geq k] = \rho_i^k \frac{G(N-k)}{G(N)}. \tag{8}$$

The expected queue length is

$$\mathrm{E}[n_i] = \sum_{k=1}^{N} (\rho_i)^k \frac{G(N-k)}{G(N)}. \tag{9}$$

These last two equations correspond to identical results shown by Buzen for a closed queuing network with no IS service station [6]. By substituting Eq. (7) for $G(N-k)$ into Eq. (9), we obtain

$$\mathrm{E}[n_i] = \frac{1}{G(N)} \sum_{l=1}^{M'} (\rho_l)^N \sum_{j=0}^{m(l)-1} \left[A_{lj} \times \sum_{n=0}^{N-1} \left(\frac{\rho_i}{\rho_l}\right)^{N-n} \binom{n+m(i)-1-j}{n} \sum_{y=0}^{N-n-1} \frac{(\rho_0/\rho_l)^y}{y!} \right]. \tag{10}$$

Example 4 All ρ_i are distinct.

$$\mathrm{E}[n_i] = \frac{1}{G(N)} \sum_{l=1}^{M} \left\{ \frac{A_l \rho_i (\rho_l)^N}{\rho_l - \rho_i} \times \sum_{n=0}^{N-1} \frac{(\rho_0/\rho_l)^n}{n!} [1 - (\rho_i/\rho_l)^{N-n}] \right\}.$$

Example 5 The queuing network has no IS service station.

$$\mathrm{E}[n_i] = \frac{1}{G(N)} \sum_{l=1}^{M} \frac{A_l \rho_i [(\rho_l)^N - (\rho_i)^N]}{\rho_l - \rho_i}.$$

- *Marginal queue length statistics for an IS service station*

The marginal queue length probability distribution for an IS service station is

$$P[n_i = k] = \frac{\rho_i^k}{k!} \frac{\tilde{G}(N-k, \rho_0 - \rho_i)}{G(N)}, \tag{11}$$

where

$$\tilde{G}(n,x) \triangleq \sum_{l=0}^{n} g(n-l) \frac{x^l}{l!}.$$

The first and second moments are

$$\mathrm{E}[n_i] = \sum_{k=1}^{N} k\, P[n_i = k] = \rho_i \frac{G(N-1)}{G(N)}; \tag{12}$$

$$\mathrm{E}[(n_i)^2] = \sum_{k=1}^{N} k^2 P[n_i = k] = [G(N)]^{-1} [(\rho_i)^2 G(N-2) + \rho_i G(N-1)]. \tag{13}$$

By replacing ρ_i by the appropriate traffic intensity, Eqs. (11)-(13) can be applied to evaluate the marginal statistics for the number of customers belonging to any combination of classes in any group of IS service stations.

Extension to semiclosed networks

For a semiclosed network, define

$$\Lambda(n) \triangleq \begin{cases} 1, & n = N_1; \\ \prod_{m=N_1}^{n-1} \lambda(m), & n > N_1. \end{cases}$$

The normalization constant has been shown by Reiser and Kobayashi [7] to be

$$C = \left[\sum_{n=N_1}^{N_2} \Lambda(n)\, G(n) \right]^{-1}.$$

Formulas for marginal queue length statistics in Eqs. (8), (9), and (11)-(13) still apply if $G(N-k)$ is replaced by

$$G'(k) \triangleq \sum_{n=\max[0,\, N_1-k]}^{N_2-k} \Lambda(n+k)\, G(n), \tag{14}$$

and $\tilde{G}(N-k,x)$ is replaced by

$$G''(k,x) \triangleq \sum_{n=\max[0,\, N_1-k]}^{N_2-k} \Lambda(n+k)\, \tilde{G}(n,x), \tag{15}$$

where $k = 0, 1, \cdots, N_2$.

Conclusion

We have extended Moore's result for a closed network of exponential servers to the case of nondistinct traffic intensities. Solutions for the normalization constant and marginal queue length statistics of various closed and semiclosed queuing networks have been shown.

Appendix

Consider M' distinct ρ_i with multiplicity $m(i)$ such that

$$\sum_{i=1}^{M'} m(i) = M.$$

Define

$$T(t) \triangleq \prod_{i=1}^{M'} [(1-\rho_i t)^{m(i)}]^{-1}. \tag{A1}$$

By expanding $T(t)$ into partial fractions,

$$T(t) = \sum_{i=1}^{M'} \sum_{j=0}^{m(i)-1} \frac{A_{ij}}{(1-\rho_i t)^{m(i)-j}}, \tag{A2}$$

where A_{ij} are constants to be determined below. From Eq. (3) and following [9], $g(n)$ is equal to the coefficient of t^n in $T(t)$, and is given by

$$g(n) = \sum_{i=1}^{M'} (\rho_i)^n \sum_{j=0}^{m(i)-1} \binom{n + m(i) - j - 1}{n} A_{ij}.$$

To solve for A_{ij}, we define

$$H(t) \triangleq \prod_{k=1,k\neq i}^{M'} [(1 - \rho_k t)^{m(k)}]^{-1}.$$

From Eqs. (A1) and (A2),

$$A_{i0} = H(t) \bigg|_{t=\frac{1}{\rho_i}} = \prod_{k=1,k\neq i}^{M'} \{[1 - (\rho_k/\rho_i)]^{m(k)}\}^{-1},$$
$$i = 1, 2, \cdots, M',$$

and

$$A_{ij} = \frac{1}{(-\rho_i)^j j!} \frac{\partial^j}{\partial t^j} H(t) \bigg|_{t=\frac{1}{\rho_i}},$$
$$1 \leq i \leq M';\ 1 \leq j \leq m(i) - 1. \qquad \text{(A3)}$$

Now A_{ij} may be found in terms of A_{il}, $l = 0, 1, \cdots, j-1$, as follows:

$$\frac{\partial}{\partial t} H(t) = \left[\sum_{k=1,k\neq i}^{M'} \frac{m(k)\rho_k}{1 - \rho_k t}\right] H(t) \triangleq F(t)\ H(t), \qquad \text{(A4)}$$

$H^{(l)} \triangleq \frac{\partial^l}{\partial t^l} H(t)$, and

$$F^{(l)} \triangleq \frac{\partial^l}{\partial t^l} F(t) = \sum_{k=1,k\neq i}^{M'} \frac{l!\ (\rho_k)^{l+1}\ m(k)}{(1 - \rho_k t)^{l+1}}. \qquad \text{(A5)}$$

From Eq. (A4),

$$H^{(j)} = \sum_{l=0}^{j-1} \binom{j-1}{l} H^{(l)}\ F^{(j-1-l)}.$$

By substituting Eq. (A3) into the above and evaluating at $t = 1/\rho_i$, we find

$$(-\rho_i)^j\ j!\ A_{ij} = \sum_{l=0}^{j-1} \binom{j-1}{l} [(-\rho_i)^l\ l!\ A_{il}]\ F^{(j-1-l)}\bigg|_{t=\frac{1}{\rho_i}}.$$

By substituting Eq. (A5) into the above and rearranging,

$$A_{ij} = \frac{1}{j} \sum_{l=0}^{j-1} A_{il}\ (-1)^{j-l} \sum_{k=1,k\neq i}^{M'} m(k) \left[\frac{\rho_k}{\rho_i - \rho_k}\right]^{j-l},$$
$$1 \leq i \leq M';\ 1 \leq j \leq m(i) - 1.$$

References

1. J. R. Jackson, "Jobshop-Like Queueing Systems," *Manage. Sci.* **10,** 131 (1963).
2. F. Baskett, K. M. Chandy, R. R. Muntz, and F. G. Palacios, "Open, Closed and Mixed Networks of Queues with Different Classes of Customers," *J. ACM* **22,** 248 (1975).
3. M. Reiser and H. Kobayashi, "Queuing Networks with Multiple Closed Chains: Theory and Computational Algorithms," *IBM J. Res. Develop.* **19,** 283 (1975).
4. H. Kobayashi and M. Reiser, "Further Results on a Queueing Network Model," *Proceedings of the Conference on Information Sciences and Systems*, Johns Hopkins University, Baltimore, MD, April 1975, p. 166.
5. S. S. Lam, "Queuing Networks with Population Size Constraints," *IBM J. Res. Develop.* **21,** 370 (1977, this issue).
6. J. P. Buzen, "Computational Algorithms for Closed Queueing Networks with Exponential Servers," *Commun. ACM* **16,** 527 (1973).
7. M. Reiser and H. Kobayashi, "Numerical Solution of Semiclosed Exponential Server Queueing Networks," *Proceedings of the Seventh Asilomar Conference on Circuits, Systems and Computers*, Monterey, CA, November 1973, p. 308.
8. R. R. Muntz and J. Wong, "Efficient Computational Procedures for Closed Queueing Networks with the Product Form Solution," *Modeling and Measurement Note No. 17*, Computer Science Dept., Univ. of California, Los Angeles, June 1973.
9. F. R. Moore, "Computational Model of a Closed Queuing Network with Exponential Servers," *IBM J. Res. Develop.* **16,** 567 (1972).

Received September 18, 1976

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