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APPENDIX A

SIMULATION RESULTS FOR THE POISSON ASSUMPTION

Channel traffic is a random variable representing the total number of packets transmitted by all users into a channel time slot. Both zeroth order and first order approximations in Chapter 3 assume that channel traffic is Poisson distributed (the Poisson assumption). In this appendix, we examine further the accuracy of the Poisson assumption through simulations.

Let P_i be the fraction of time slots, each of which has exactly i packet transmissions, over the duration of a simulation run. $\{P_i\}_{i=0}^{M}$ represents the measured probability distribution for channel traffic. (M is the number of channel users.) The channel throughput rate S_{out} is given by P_1. The channel traffic rate G is given by $\sum_{i=1}^{M} i\, P_i$. We give below comparisons between P_i and the Poisson probabilities $\frac{G^i}{i!} e^{-G}$ for the infinite population model, the linear feedback model and controlled channels.

In Table A.1, P_i and the corresponding Poisson probabilities are shown for various cases of the infinite population model. In all cases, $R = 12$ and the simulation duration is 8000 time slots. Each simulation run satisfies the channel equilibrium criterion in Section 3.2.3. Cases (a), (b) and (c) correspond to $K = 5$, 15 and 40 respectively with $S_{out} \cong 0.25$. Note that the Poisson approximation is better for $K = 15$, 40 than $K = 5$. (This observation is consistent with the conclusion of Theorem 4.1.) Cases (b), (d) and (e) correspond to S_{out} = 0.245, 0.150 and 0.304 respectively with $K = 15$. Note that the Poisson approximation is better for a smaller S_{out}.

In Table A.2, comparisons are shown for the linear feedback model with $M = 200$ and four different retransmission delay probability distributions (corresponding to those in Fig. 5-1). Each simulation run has a duration of 8000 time slots and satisfies the channel equilibrium criterion. In all four cases, the Poisson approximation is excellent.

In Table A.3, comparisons are shown for three controlled channels with $M = 200$: (a) ICP-CONTEST with $W = 40$ and $\hat{n} = 22$, (b) RCP-CONTEST with $W = 40$ and $\hat{n} = 18$, and (c) Heuristic RCP with $K_1 = 10$ and $K_m = 60$ for $m \geq 2$. R is assumed to be 12 and each simulation run has a duration of 30,000 time slots. In all cases the Poisson approximation is quite good. (Note that performance of the CONTEST algorithms depends upon the accuracy of $P_0 \cong e^{-G}$ within a time history window.)

From comparisons in Tables A.1 - A.3, we also observe the following:

(1) In all cases, $P_0 \geq e^{-G}$

(2) In all cases, $P_1 \leq Ge^{-G}$; this is expected since finite retransmission delays are used.

(3) In most cases, P_i $(2 \leq i \leq 6)$ are larger than the corresponding Poisson probabilities. On the other hand, the Poisson distribution has a much longer tail than the measured channel traffic probability distribution.

(a) $K = 5$

$S_{out} = 0.253$

$G = 0.432$

i	P_i	Poisson
0	0.6671	0.6490
1	0.2530	0.2806
2	0.0649	0.0607
3	0.0116	0.0087
4	0.0025	0.0009
5	0.0005	0.0001
6	0.0004	0.0000

(b) $K = 15$

$S_{out} = 0.245$

$G = 0.361$

i	P_i	Poisson
0	0.7011	0.6973
1	0.2451	0.2514
2	0.0466	0.0453
3	0.0064	0.0054
4	0.0008	0.0005
5	0.0000	0.0000
6	0.0000	0.0000

(c) $K = 40$

$S_{out} = 0.252$

$G = 0.384$

i	P_i	Poisson
0	0.6872	0.6814
1	0.2516	0.2614
2	0.0524	0.0501
3	0.0080	0.0064
4	0.0008	0.0006
5	0.0000	0.0000
6	0.0000	0.0000

Table A.1 Channel traffic probability distribution (infinite population model).

(d) $K = 15$

$S_{out} = 0.150$

$G = 0.184$

i	P_i	Poisson
0	0.8335	0.8315
1	0.1500	0.1534
2	0.0153	0.0142
3	0.0011	0.0009
4	0.0000	0.0000
5	0.0001	0.0000
6	0.0000	0.0000

(e) $K = 15$

$S_{out} = 0.304$

$G = 0.586$

i	P_i	Poisson
0	0.5722	0.5563
1	0.3045	0.3263
2	0.0946	0.0957
3	0.0229	0.0187
4	0.0048	0.0027
5	0.0009	0.0003
6	0.0001	0.0000

Table A.1 (Continued)

i	P_i	Poisson
0	0.6086	0.6021
1	0.2944	0.3055
2	0.0805	0.0775
3	0.0141	0.0131
4	0.0023	0.0016
5	0.0001	0.0002
6	0.0000	0.0000

(a) $R = 12 \quad K = 10$

$S_{out} = 0.294$

$G = 0.507$

i	P_i	Poisson
0	0.6264	0.6256
1	0.2934	0.2934
2	0.0670	0.0688
3	0.0116	0.0108
4	0.0014	0.0013
5	0.0002	0.0001
6	0.0000	0.0000

(b) $R = 0 \quad K = 34$

$S_{out} = 0.293$

$G = 0.469$

i	P_i	Poisson
0	0.6308	0.6279
1	0.2894	0.2922
2	0.0661	0.0680
3	0.0113	0.0105
4	0.0023	0.0012
5	0.0001	0.0001
6	0.0000	0.0000

(c) $R = 12 \quad p = \frac{2}{11}$

$S_{out} = 0.289$

$G = 0.465$

i	P_i	Poisson
0	0.6373	0.6351
1	0.2831	0.2883
2	0.0691	0.0655
3	0.0095	0.0099
4	0.0009	0.0011
5	0.0001	0.0001
6	0.0000	0.0000

(d) $R = 0 \quad p = \frac{2}{35}$

$S_{out} = 0.283$

$G = 0.454$

Table A.2 Channel traffic probability distribution (M = 200).

(a) ICP-CONTEST

$K = 10$

$S_{out} = 0.315$

$G = 0.597$

i	P_i	Poisson
0	0.5612	0.5505
1	0.3148	0.3286
2	0.0963	0.0981
3	0.0223	0.0195
4	0.0044	0.0029
5	0.0009	0.0004
6	0.0001	0.0000

(b) RCP-CONTEST

$K_o = 10 \quad K_c = 60$

$S_{out} = 0.322$

$G = 0.655$

i	P_i	Poisson
0	0.5340	0.5193
1	0.3218	0.3403
2	0.1084	0.1115
3	0.0282	0.0243
4	0.0061	0.0040
5	0.0014	0.0005
6	0.0000	0.0001

(c) Heuristic RCP

$K_1 = 10 \quad K_2 = 60$

$S_{out} = 0.316$

$G = 0.579$

i	P_i	Poisson
0	0.5670	0.5605
1	0.3163	0.3245
2	0.0922	0.0939
3	0.0205	0.0181
4	0.0034	0.0026
5	0.0005	0.0003
6	0.0001	0.0000

Table A.3 Channel traffic probability distribution (Controlled Channels).

APPENDIX B

ANALYSIS FOR THE LARGE USER MODEL

The set of nonlinear implicit equations involving equilibrium values of S_i, G_i, q_{in} and q_{it} (i = 1, 2) in the large user model will be derived. Recall that variables indexed by 1 refer to the small users and variables indexed by 2 refer to the large user.

Define E_1 and E_2 to be the average number of channel collisions for a small user and a large user packet respectively. Similar to the derivation of Eq. (3.5), we have

$$E_i = (1 - q_{in})/q_{it} \qquad i = 1, 2 \tag{B.1}$$

$$G_i = S_i(1 + E_i) \qquad i = 1, 2 \tag{B.2}$$

Thus

$$S_i = G_i \frac{q_{it}}{q_{it} + 1 - q_{in}} \qquad i = 1, 2 \tag{B.3}$$

which are Eqs. (3.16) and(3.17).

Referring to the model description of a large user in Section 2.3.2, we introduce the following notation for events at the large user:

TS = transmission success in a channel slot

SS = scheduling success (i.e., capture of the transmitter as a result of having the highest priority among all packets scheduled for the current time slot)

Each large user packet may be in one of the following three states depending on their most recent history:

NP = newly generated packet

SC = scheduling conflict (i.e., failure to capture transmitter)

TC = transmission conflict in a channel slot

Now define the variables,

a_n = Prob [SS/NP]

a_t = Prob [SS/TC]

a_s = Prob [SS/SC]

r_n = Prob [TS/SS, NP]

r_t = Prob [TS/SS, TC]

r_s = Prob [TS/SS, SC]

Given a large user packet, let E_n and E_t be the average number of SC events before SS, conditioning on NP and TC respectively. Similar to the derivation of Eq. (3.5), we have

$$\begin{aligned} E_n &= (1 - a_n)/a_s \\ E_t &= (1 - a_t)/a_s \end{aligned} \tag{B.4}$$

Recalling the definitions of q_{2n} and q_{2t}, we have

$$\begin{aligned} q_{2n} &= (r_n + r_s E_n)/(1 + E_n) \\ q_{2t} &= (r_t + r_s E_t)/(1 + E_t) \end{aligned} \tag{B.5}$$

The average station traffic (defined in Section 3.3.2) is

$$G_s = S_2 \; [1 + E_n + E_2(1 + E_t)] \tag{B.6}$$

and the average packet delays are

$$D_1 = R + 1 + E_1\left[R + \frac{K+1}{2}\right] \tag{3.18}$$

$$D_2 = R + 1 + E_2\left[R + \frac{K+1}{2}\right] + (E_n + E_2E_t)\frac{L+1}{2} \tag{3.19}$$

where $R + \frac{K+1}{2}$ is the average retransmission delay and $\frac{L+1}{2}$ is the average reschedule delay (see Section 2.3.2).

With the Poisson and independence assumptions in Section 3.3.2 for channel traffic and station traffic, we proceed to solve for the success probabilities q_1, q_{1t}, r_n, r_t, r_s, a_n, a_t and a_s. (The approach is similar to the derivation of q_n and q_t in the infinite population model.) Consider the transmission of a test packet in the current time slot; a conflict may occur as a result of new arrivals, packets retransmitting from a window of K slots or packets rescheduling from a window of L slots in the past.

Define

q_o = Prob [no packet retransmitting from one of the K slots to the current slot]

and

q_h = Prob [no packet rescheduling from one of the L slots to the current slot]

We then have,

$$q_o = \sum_{n,m\geq 1} \frac{G_1^n}{n!} e^{-G_1} \frac{G_s^m}{m!} e^{-G_s}\left(\frac{K-1}{K}\right)^{n+1} + e^{-G_1}\sum_{m\geq 1}\frac{G_s^m}{m!} e^{-G_s} +$$

$$e^{-G_s}\sum_{n\geq 2}\frac{G_1^{\,n}}{n!}e^{-G_1}\left(\frac{K-1}{K}\right)^n + G_1 e^{-(G_1+G_s)} + e^{-(G_1+G_s)}$$

$$= e^{-G_1/K} + \frac{1}{K}\left[\left(1-e^{-G_s}\right)\left(e^{-G_1}-e^{-G_1/K}\right) + G_1 e^{-(G_1+G_s)}\right]$$

$$q_h = \sum_{m\geq 2}\frac{G_s^{\,m}}{m!}e^{-G_s}\left(\frac{L-1}{L}\right)^{m-1} + G_s e^{-G_s} + e^{-G_s}$$

$$= \left[L\, e^{-G_s/L} - e^{-G_s}\right]\Big/(L-1) \qquad L\geq 2$$

$$q_h = (G_s+1)\, e^{-G_s} \qquad L = 1$$

Suppose the test packet is a small user packet. Conditioning on a new packet, we have

$$q_{1n} = q_o^{\,K} q_h^{\,L} e^{-S} \tag{B.7}$$

Conditioning on a packet which had a channel collision in the j^{th} slot, define

q_{1c} = Prob [no other packet retransmitting from the j^{th} slot to the current slot]

$$q_{1c} = \frac{1}{1-e^{-(G_1+G_s)}}\left[(1-e^{-G_s})\sum_{n\geq 0}\frac{G_1^{\,n}}{n!}e^{-G_1}\left(\frac{K-1}{K}\right)^{n+1} + e^{-G_s}\sum_{n\geq 1}\frac{G_1^{\,n}}{n!}e^{-G_1}\left(\frac{K-1}{K}\right)^{n}\right]$$

$$= \left[e^{-G_1/K}\left(1-\frac{1-e^{-G_s}}{K}\right) - e^{-(G_1+G_s)}\right]\Big/\left[1-e^{-(G_1+G_s)}\right]$$

We have,

$$q_{1t} = q_o^{K-1} q_{1c} q_h^{L} e^{-S} \tag{B.8}$$

Suppose the test packet is a large user packet and condition on the event SS. Define

$\bar{q}$ = Prob [no small user packet retransmitting from one of the K slots to the current slot]

$$\bar{q} = \sum_{m+n\geq 2} \frac{G_1^{\,n}}{n!} e^{-G_1} \frac{G_s^{\,m}}{m!} e^{-G_s} \left(\frac{K-1}{K}\right)^n + (G_1+G_s)\, e^{-(G_1+G_s)} + e^{-(G_1+G_s)}$$

$$= e^{-G_1/K} + \frac{G_1}{K} e^{-(G_1+G_s)}$$

Conditioning on the event NP, we have

$$r_n = \text{Prob [TS/SS, NP]} = \bar{q}^K e^{-S_1} \tag{B.9}$$

and

$$\text{Prob [SS, TS/NP]} = q_o^{K} q_h^{L} e^{-S_1} (1-e^{-S_2})/S_2$$

where we have made use of the scheduling priority rule in Section 2.3.2 in which new packets have the lowest priority; ties among new packets are broken by random selection such that

$$(1-e^{-S_2})/S_2 = \sum_{i=0}^{\infty} \frac{S_2^{\,i}}{i!} e^{-S_2} \left(\frac{1}{i+1}\right)$$

Finally,

$$a_n = \text{Prob}\ [SS/NP]$$

$$= \text{Prob}\,[SS,\ TS/NP] \big/ \text{Prob}\,[TS/SS,\ NP]$$

$$= \left(q_o/\bar{q}\right)^K q_h^{\ L}\ \left(1-e^{-S_2}\right) \Big/ S_2 \qquad \text{(B.10)}$$

Given that the large user test packet had a channel collision (TC) in the j^{th} slot, define

q_{2c} = Prob [no small user packet retransmitting from the j^{th} slot to the current slot]

$$q_{2c} = \left[\sum_{n\geq 1} \frac{G_1^{\ n}}{n!} e^{-G_1} \left(\frac{K-1}{K}\right)^n\right] \Big/ \left[1-e^{-G_1}\right]$$

$$= \left[e^{-G_1/K} - e^{-G_1}\right] \Big/ \left[1-e^{-G_1}\right]$$

We then have

$$r_t = \text{Prob}\ [TS/SS,\ TC]$$

$$= \bar{q}^{\,K-1}\ q_{2c}\ e^{-S_1} \qquad \text{(B.11)}$$

and

$$\text{Prob}\ [SS,\ TS/TC] = \sum_{i=1}^{K} \frac{1}{K}\, q_o^{\ i-1} q_{2c}\ \bar{q}^{\,K-i} e^{-S_1}$$

where the scheduling priority rule has been used.

Finally

$$a_t = \text{Prob } [SS/TC]$$

$$= \text{Prob}[SS,\ TS/TC] / \text{Prob}[TS/SS,\ TC]$$

$$= \frac{1}{K}\left[1 - \left(q_o/\bar{q}\right)^K\right] \Big/ \left[1 - \left(q_o/\bar{q}\right)\right] \tag{B.12}$$

Given that the large user packet had a scheduling conflict (SC) in the j^{th} slot, define

q_{sc} = Prob [no other packet rescheduling from the j^{th} slot to the current slot]

$$q_{sc} = \left[\sum_{m=1}^{\infty} \frac{G_s^{\ m}}{m!} e^{-G_s} \frac{m}{m+1} \left(\frac{L-1}{L}\right)^{m-1}\right] \Bigg/ \left[\sum_{m=1}^{\infty} \frac{G_s^{\ m}}{m!} e^{-G_s} \frac{m}{m+1}\right]$$

$$= \left(\frac{L}{L-1}\right)^2 \left[\frac{G_s\left(1 - \frac{1}{L}\right) e^{-G_s/L} - e^{-G_s/L} + e^{-G_s}}{G_s - 1 + e^{-G_s}}\right]$$

We have

$$r_s = \text{Prob } [TS/SS,\ SC] \tag{B.13}$$

$$= \bar{q}^K e^{-S_1}$$

Prob [SS, TS/SC]

$$= q_o^{\ K}\, q_{sc} \sum_{i=1}^{L} \frac{1}{L}\, q_h^{\ i-1} e^{-S_1}$$

where the scheduling priority rule has been used. Finally

$$a_s = \text{Prob } [SS/SC]$$

$$= \text{Prob}[SS,\ TS/SC]/\text{Prob}[TS/SS,\ SC]$$

$$= \left(q_o/\bar{q}\right)^K \frac{q_{sc}}{L} \frac{1-q_h^{\ L}}{1-q_h} \tag{B.14}$$

Eqs. (B.3) - (B.14) constitute a set of nonlinear implicit equations which may be solved numerically with specified values of K, L, G_1, and G_2 (or S_1 and S_2).

<u>Limiting results</u>

In the limit as K, L $\rightarrow \infty$, the following limiting values may be obtained from the definitions of q_o, q_h, $\bar{q}$, q_{1c}, q_{2c} and q_{sc}:

$$q_o^{\ K} = e^{-G_1\left(1-e^{-(G_1+G_s)}\right) - \left(1-e^{-G_1}\right)\left(1-e^{-G_s}\right)}$$

$$q_h^{\ L} = e^{-G_s+1-e^{-G_s}}$$

$$\bar{q}^K = e^{-G_1\left(1-e^{-(G_1+G_s)}\right)}$$

$$q_{1c} = q_{2c} = q_{sc} = 1$$

With the above limiting results, the following proposition may be shown.

Proposition B.1 In the limit as $K, L \to \infty$,

$$q_{1n} = q_{1t} = e^{-G_1}(1-G_2) \tag{3.20}$$

$$S_1 = G_1 e^{-G_1}(1-G_2) \tag{3.21}$$

$$q_{2n} = q_{2t} = e^{-G_1} \tag{3.22}$$

$$S_2 = G_2 e^{-G_1} \tag{3.23}$$

$$G_2 = 1 - e^{-G_s} \tag{3.24}$$

$$r_n = r_t = r_s = e^{-G_1}$$

$$a_n = e^{-(G_s-S_2)}\left[1 - e^{-S_2}\right] \Big/ S_2$$

$$a_t = \left[1 - e^{-(G_2-S_2)}\right] \Big/ (G_2-S_2)$$

$$a_s = e^{-(G_2-S_2)}\left[1 - e^{-(G_s-G_2)}\right] \Big/ (G_s-G_2)$$

Proof The variables in the above equations are defined by Eqs. (B.3) - (B.14). It suffices to show that limiting values of these variables given by the proposition satisfy Eqs. (B.3) - (B.14) in the K, $L \to \infty$ limit. This may be accomplished by assuming the proposition to be true, evaluating the RHSs of Eqs. (B.3) - (B.14) and showing that they are equal to the corresponding LHSs in the K, $L \to \infty$ limit.

APPENDIX C

DERIVATION OF EQS. (4.3) AND (4.4), THEOREM 4.1 AND ITS PROOF

Derivation of Eq. (4.3)

By definition,

$$Q^{t+1}(\underline{z}) = \sum_{y_1=0}^{\infty} \cdots \sum_{y_{R+K}=0}^{\infty} \left(\prod_{j=1}^{R+K} z_j^{\,y_j} \right) P^{t+1}(\underline{y})$$

Substituting Eqs. (4.1) and (4.2) for $P^{t+1}(\underline{y})$, we have

$$Q^{t+1}(\underline{z}) = \sum_{y_1=0}^{\infty} \cdots \sum_{y_{R+K}=0}^{\infty} \left(\prod_{j=1}^{R+K} z_j^{\,y_j} \right) \sum_{x_{R+K}=0}^{\infty} \sum_{\substack{i=0 \\ i\le \ell}}^{y_1} v_{y_1-i}^{t+1} \binom{\ell}{i} \left(\frac{1}{K}\right)^{i} \left(1-\frac{1}{K}\right)^{\ell-i} P^{t}(\underline{x})$$

where $x_i = y_{i+1}$ for i = 1, 2, ..., R+K-1

$$\lambda(m) = \begin{cases} 0 & m = 1 \\ m & m \neq 1 \end{cases}$$

and

$$\ell = \sum_{j=1}^{K} \lambda(x_{R+j})$$

Exchanging the order of the first and last summations, and evaluating their sum,

$$Q^{t+1}(\underline{z}) = \sum_{y_2=0}^{\infty} \cdots \sum_{y_{R+K}=0}^{\infty} \left(\prod_{j=2}^{R+K} z_j^{\,y_j} \right) \sum_{x_{R+K}=0}^{\infty} V^{t+1}(z_1) \left[1 - \frac{1}{K} + \frac{z_1}{K} \right]^{\ell} P^t(\underline{x})$$

Letting $y_i = x_{i-1}$ for $i = 2, 3, \ldots, R+K$ and rearranging,

$$Q^{t+1}(\underline{z}) = V^{t+1}(z_1) \sum_{x_1=0}^{\infty} \cdots \sum_{x_{R+K}=0}^{\infty} \left(\prod_{j=1}^{R+K-1} z_{j+1}^{\,x_j} \right) \left[1 - \frac{1}{K} + \frac{z_1}{K} \right]^{\ell} P^t(\underline{x})$$

$$= V^{t+1}(z_1) \sum_{x_1=0}^{\infty} \cdots \sum_{x_{R+K}=0}^{\infty} \left(\prod_{j=1}^{R} z_{j+1}^{\,x_j} \right) \left(\prod_{j=R+1}^{R+K-1} z_{j+1}^{\,x_j} \left[1 - \frac{1}{K} + \frac{z_1}{K} \right]^{\lambda(x_j)} \right) \cdot$$

$$\left[1 - \frac{1}{K} + \frac{z_1}{K} \right]^{\lambda(x_{R+K})} \cdot P^t(\underline{x}) \qquad \text{(C.1)}$$

which is given by Eq. (4.3) and its accompanying algorithm.

Derivation of Eq. (4.4)

Define

$h_i^{\,t-R-j}$ = Prob[exactly i packets retransmitting from the $(t-R-j)^{th}$ slot to the t^{th} slot]

We then have,

$$
h_i^{t-R-j} = \begin{cases} P_0^{t-R-j} + P_1^{t-R-j} + \sum_{m=2}^{\infty}\left(1 - \frac{1}{K}\right)^m P_m^{t-R-j} & i = 0 \\ \sum_{m=2}^{\infty}\binom{m}{1}\left(\frac{1}{K}\right)\left(1 - \frac{1}{K}\right)^{m-1} P_m^{t-R-j} & i = 1 \\ \sum_{m=i}^{\infty}\binom{m}{i}\left(\frac{1}{K}\right)^i \left(1 - \frac{1}{K}\right)^{m-i} P_m^{t-R-j} & i \geq 2 \end{cases} \tag{C.2}
$$

Now define

$$
\hat{Q}^{t-R-j}(z) = \sum_{i=0}^{\infty} z^i h_i^{t-R-j}
$$

Substituting Eqs. (C.2) into the above equation and summing, we get

$$
\begin{aligned} \hat{Q}^{t-R-j}(z) &= P_1^{t-R-j} \frac{(1-z)}{K} + \sum_{m=0}^{\infty}\left(1 - \frac{1}{K} + \frac{z}{K}\right)^m P_m^{t-R-j} \\ &= P_1^{t-R-j} \frac{(1-z)}{K} + Q^{t-R-j}\left(1 - \frac{1}{K} + \frac{z}{K}\right) \end{aligned} \tag{C.3}
$$

Finally, by the weak independence assumption for channel traffic and the assumption that the channel input V^t is independent of the channel state,

$$
Q^t(z) = V^t(z) \prod_{j=1}^{K} \hat{Q}^{t-R-j}(z) \tag{C.4}
$$

which is the same as Eq. (4.4).

Theorem 4.1 and its proof

Theorem 4.1 If the channel input is an independent Poisson process, then the channel traffic is Poisson distributed in the limit as $K \to \infty$ under the weak independence assumption, such that

$$Q^t(z) = e^{-G^t(1-z)}$$

and

$$P_1{}^t = G^t e^{-G^t}$$

where

$$G^t = \frac{1}{K}\sum_{j=1}^{K}\left(G^{t-R-j} - G^{t-R-j}e^{-G^{t-R-j}}\right) + S^t$$

Proof Since V^t has a Poisson distribution,

$$V^t(z) = e^{-S^t(1-z)}$$

Substituting it into Eq. (4.4), we have

$$Q^t(z) = e^{-S^t(1-z)} \prod_{j=1}^{K}\left[Q^{t-R-j}\left(1 - \frac{1}{K} + \frac{z}{K}\right) + P_1{}^{t-R-j}\ \frac{1-z}{K}\right] \tag{C.5}$$

Consider

$$Q^{t-R-j}\left(1 - \frac{1}{K} + \frac{z}{K}\right) = P_0{}^{t-R-j} + \sum_{i=1}^{\infty} P_i{}^{t-R-j}\left[1 - \frac{i}{K}(1-z)\right] + o\left(\frac{1}{K}\right)$$

$$= 1 - G^{t-R-j}\,\frac{(1-z)}{K} + o\left(\frac{1}{K}\right) \tag{C.6}$$

where

$$\lim_{x \to 0} \frac{o(x)}{x} \to 0$$

Substituting Eq. (C.6) into Eq. (C.5) and letting $K \to \infty$

$$\lim_{K\to\infty} Q^t(z) = e^{-S^t(1-z)} \prod_{j=1}^{K} \left[1 - \left(G^{t-R-j} - P_1^{\ t-R-j} \right) \frac{(1-z)}{K} + o\left(\frac{1}{K}\right) \right]$$

$$= e^{-\left[\frac{1}{K} \sum_{j=1}^{K} \left(G^{t-R-j} - P_1^{\ t-R-j} \right) + S^t \right](1-z)}$$

$$= e^{-G^t (1-z)} \qquad \text{(C.7)}$$

where

$$G^t = \frac{1}{K} \sum_{j=1}^{K} \left(G^{t-R-j} - P_1^{\ t-R-j} \right) + S^t \qquad \text{(C.8)}$$

From Eq. (C.7), we get

$$\lim_{K\to\infty} P_1^{\ t} = G^t e^{-G^t} \qquad \text{(C.9)}$$

Q.E.D.

APPENDIX D

ALGORITHM 5.1, ITS DERIVATION AND

SOME MONOTONE PROPERTIES

Algorithm 5.1

This algorithm solves for the variables $\{t_i\}_{i=0}^{I}$ in the following set of $(I + 1)$ linear simultaneous equations,

$$t_0 = h_0 + \sum_{j=0}^{I} p_{0j} \, t_j \tag{D.1}$$

$$t_i = h_i + \sum_{j=i-1}^{I} p_{ij} \, t_j \qquad i = 1, 2, \ldots, I \tag{D.2}$$

(1) Define

$$e_I = 1$$

$$f_I = 0$$

$$e_{I-1} = \frac{1 - p_{II}}{p_{I,I-1}}$$

$$f_{I-1} = - \frac{h_I}{p_{I,I-1}}$$

(2) For $i = I - 1, I - 2, \ldots, 1$ solve recursively

$$e_{i-1} = \frac{1}{p_{i,i-1}} \left[e_i - \sum_{j=i}^{I} p_{ij} \, e_j \right]$$

$$f_{i-1} = \frac{1}{p_{i,i-1}} \left[f_i - h_i - \sum_{j=i}^{I} p_{ij} \, f_j \right]$$

(3) Let

$$t_I = \frac{f_0 - h_0 - \sum_{j=0}^{I} p_{0j} \, f_j}{\sum_{j=0}^{I} p_{0j} \, e_j - e_0}$$

$$t_i = e_i \, t_I + f_i \qquad i = 0, 1, 2, \ldots, I - 1$$

<u>Derivation of Algorithm 5.1</u>

Define

$$t_i = e_i \, t_I + f_i \qquad i = 0, 1, 2, \ldots, I - 1 \tag{D.3}$$

and

$$e_I = 1$$

$$f_I = 0 \tag{D.4}$$

The last equation in Eqs. (D.2) is

$$t_I = h_I + p_{I,I-1}\, t_{I-1} + p_{II}\, t_I$$

Substituting $t_{I-1} = e_{I-1}\, t_I + f_{I-1}$ into the above equation, we get

$$t_I = h_I + p_{I,I-1}\, e_{I-1}\, t_I + p_{I,I-1}\, f_{I-1} + p_{II}\, t_I$$

Equating the coefficients of t_I and the constant terms, we have

$$e_{I-1} = \frac{1 - p_{II}}{p_{I,I-1}}$$

$$f_{I-1} = -\frac{h_I}{p_{I,I-1}} \qquad \text{(D.5)}$$

Eqs. (D.2) can be rewritten as follows,

$$t_{i-1} = \frac{1}{p_{i,i-1}} \left[t_i - h_i - \sum_{j=i}^{I} p_{ij}\, t_j \right] \qquad \text{(D.6)}$$

In each of the above equations, use Eqs. (D.3) to substitute for t_i . We then have

$$e_{i-1}\, t_I + f_{i-1} = \frac{1}{p_{i,i-1}} \left[e_i\, t_I + f_i - h_i - \left(\sum_{j=i}^{I} p_{ij}\, e_j \right) t_I - \sum_{j=i}^{I} p_{ij}\, f_j \right]$$

Equating the coefficients of t_I and the constant terms, we get

$$e_{i-1} = \frac{1}{p_{i,i-1}} \left[e_i - \sum_{j=i}^{I} p_{ij} \, e_j \right]$$

$$f_{i-1} = \frac{1}{p_{i,i-1}} \left[f_i - h_i - \sum_{j=i}^{I} p_{ij} \, f_j \right] \tag{D.7}$$

From Eqs. (D.4), (D.5) and (D.7), e_i and f_i $(i = I-2, I-3, \ldots, 1, 0)$ can then be determined recursively.

We next solve for t_I. Eqs. (D.3) are used to substitute for t_i in Eq. (D.1), which then becomes

$$e_0 \, t_I + f_0 = h_0 + \left(\sum_{j=0}^{I} p_{0j} \, e_j \right) t_I + \sum_{j=0}^{I} p_{0j} \, f_j$$

Solving for t_I in the above equation, we have

$$t_I = \frac{f_0 - h_0 - \sum_{j=0}^{I} p_{0j} \, f_j}{\sum_{j=0}^{I} p_{0j} \, e_j - e_0} \tag{D.8}$$

Finally, $t_i (i = 0, 1, 2, \ldots, I - 1)$ can be obtained from Eqs. (D.3), since e_i, f_i and t_I are all known. The derivation of Algorithm 5.1 is complete.

Some monotone properties

We show below monotone properties of the sequences e_i and f_i in Algorithm 5.1. The transition probabilities p_{ij} are assumed to be nonnegative and for each $i = 1, 2, \ldots$

$$\sum_{j=i-1}^{\infty} p_{ij} = 1$$

Also, the probabilities $p_{i,i-1}$ are assumed to be nonzero. (This last is a necessary condition for the Markov process in Section 5.1.1 to be irreducible.)

Property D.1 The sequence e_i is positive and monotonically decreases to one as i increases to I.

Proof From Eqs. (D.4) and (D.5),

$$e_I = 1$$

$$e_{I-1} = \frac{1 - p_{II}}{p_{I,I-1}} > 1$$

The proof is by induction. Assume that e_ℓ decreases as ℓ increases for $i \le \ell \le I$. From Eqs. (D.7)

$$e_{i-1} = \frac{1}{p_{i,i-1}} \left[e_i - \sum_{j=i}^{I} p_{ij} e_j \right]$$

$$> \frac{e_i \left[1 - \sum_{j=i}^{I} p_{ij} \right]}{p_{i,i-1}} > e_i$$

Q.E.D.

<u>Property D.2</u> (i) If $h_i > 0$, then the sequence f_i is negative and monotonically increases to zero as i increases to I . (ii) If $h_i < 0$, the sequence f_i is positive and monotonically decreases to zero as i increases to I .

<u>Proof</u> (i) From Eqs. (D.4) and (D.5),

$$f_I = 0$$

$$f_{I-1} = -\frac{h_I}{p_{I,I-1}}$$

The proof is by induction. Assume that f_ℓ increases as ℓ increases for $i \le \ell \le I$. From Eqs. (D.7)

$$f_{i-1} = \frac{1}{p_{i,i-1}} \left[f_i - h_i - \sum_{j=i}^{I} p_{ij} f_j \right]$$

$$< \frac{f_i \left[1 - \sum_{j=i}^{I} p_{ij} \right]}{p_{i,i-1}} < f_i$$

(ii) The proof is similar to that of (i) .

Q.E.D.

APPENDIX E

ALGORITHM 6.5, ITS DERIVATION AND SOME MONOTONE PROPERTIES

Algorithm 6.5

This algorithm solves for g and $\{v_i\}_{i=1}^{M}$ in the following set of $(M + 1)$ linear simultaneous equations,

$$g = C_0 + \sum_{j=1}^{M} p_{0j} v_j \tag{E.1}$$

$$g + v_1 = C_1 + \sum_{j=1}^{M} p_{1j} v_j \tag{E.2}$$

$$g + v_i = C_i + \sum_{j=i-1}^{M} p_{ij} v_j \qquad i = 2, 3, \ldots, M \tag{E.3}$$

where

$$\sum_{j=0}^{M} p_{0j} = \sum_{j=i-1}^{M} p_{ij} = 1 \qquad i = 1, 2, \ldots, M \tag{E.4}$$

(1) Define

$$b_{M-1} = \frac{1}{p_{M,M-1}}$$

$$d_{M-1} = -\frac{C_M}{p_{M,M-1}}$$

(2) For $i = M - 1, M - 2, \ldots, 2$ solve recursively

$$b_{i-1} = \frac{1}{p_{i,i-1}} \left[b_i + 1 - \sum_{j=i}^{M-1} p_{ij} \, b_j \right]$$

$$d_{i-1} = \frac{1}{p_{i,i-1}} \left[d_i - C_i - \sum_{j=i}^{M-1} p_{ij} \, d_j \right]$$

(3) Define

$$u_M = - \frac{1}{p_{10}} \left[b_1 + 1 - \sum_{j=1}^{M-1} p_{1j} \, b_j \right]$$

$$w_M = - \frac{1}{p_{10}} \left[d_1 - C_1 - \sum_{j=1}^{M-1} p_{1j} \, d_j \right]$$

$$u_i = u_M + b_i \qquad i = 1, 2, \ldots, M - 1$$

$$w_i = w_M + d_i$$

(4) Let

$$g = \frac{C_0 + \sum_{j=1}^{M} p_{0j} \, w_j}{1 - \sum_{j=1}^{M} p_{0j} \, u_j}$$

$$v_i = u_i \, g + w_i \qquad i = 1, 2, \ldots, M$$

Derivation of Algorithm 6.5

Define

$$v_i = u_i\, g + w_i \qquad i = 1, 2, \ldots, M \tag{E.5}$$

The above equations are substituted into Eqs. (E.2) and (E.3) for the variables v_i. Equating the coefficients of g and the constant terms in the resulting equations, we obtain two sets of M linear simultaneous equations in terms of $\{u_i\}_{i=1}^{M}$ and $\{w_i\}_{i=1}^{M}$:

$$u_1 = -1 + \sum_{j=1}^{M} p_{1j}\, u_j$$

$$u_i = -1 + \sum_{j=i-1}^{M} p_{ij}\, u_j \qquad i = 2, 3, \ldots, M \tag{E.6}$$

and

$$w_1 = C_1 + \sum_{j=1}^{M} p_{1j}\, w_j$$

$$w_i = C_i + \sum_{j=i-1}^{M} p_{ij}\, w_j \qquad i = 2, 3, \ldots, M \tag{E.7}$$

Applying Algorithm 5.1 to Eqs. (E.6), we have

$$u_i = e_i \, u_M + b_i \qquad i = 1, 2, \ldots, M - 1 \tag{E.8}$$

and

$$u_M = \frac{b_1 + 1 - \sum_{j=1}^{M} p_{1j} \, b_j}{\sum_{j=1}^{M} p_{1j} \, e_j - e_1} \tag{E.9}$$

where we define

$$e_M = 1$$

$$b_M = 0 \tag{E.10}$$

$$e_{M-1} = \frac{1 - p_{MM}}{p_{M,M-1}}$$

$$b_{M-1} = \frac{1}{p_{M,M-1}} \tag{E.11}$$

and for $i = M - 1, M - 2, \ldots, 2$ we solve recursively

$$e_{i-1} = \frac{1}{p_{i,i-1}} \left[e_i - \sum_{j=i}^{M} p_{ij} \, e_j \right]$$

$$b_{i-1} = \frac{1}{p_{i,i-1}} \left[b_i + 1 - \sum_{j=i}^{M} p_{ij} \, b_j \right] \tag{E.12}$$

Similarly, applying Algorithm 5.1 to Eqs. (E.7), we have

$$w_i = f_i \, w_M + d_i \qquad i = 1, 2, \ldots, M - 1 \tag{E.13}$$

and

$$w_M = \frac{d_1 - C_1 - \sum_{j=1}^{M} p_{1j} \, d_j}{\sum_{j=1}^{M} p_{1j} \, f_j - f_1} \tag{E.14}$$

where we define

$$\begin{aligned} f_M &= 1 \\ d_M &= 0 \end{aligned} \tag{E.15}$$

$$\begin{aligned} f_{M-1} &= \frac{1 - p_{MM}}{p_{M,M-1}} \\ d_{M-1} &= - \frac{C_M}{p_{M,M-1}} \end{aligned} \tag{E.16}$$

and for $i = M - 1, M - 2, \ldots, 2$ we solve recursively

$$\begin{aligned} f_{i-1} &= \frac{1}{p_{i,i-1}} \left[f_i - \sum_{j=i}^{M} p_{ij} \, f_j \right] \\ d_{i-1} &= \frac{1}{p_{i,i-1}} \left[d_i - C_i - \sum_{j=i}^{M} p_{ij} \, d_j \right] \end{aligned} \tag{E.17}$$

We note from Eqs. (E.10)-(E.12) and Eqs. (E.15)-(E.17) that $e_i = f_i$ for $i = 1, 2, \ldots, M$. We proceed to show that $e_i = f_i = 1$ for all i . From Eqs. (E.10) and (E.11)

$$e_M = 1$$

$$e_{M-1} = \frac{1 - p_{MM}}{p_{M,M-1}} = 1$$

This last is true by virtue of Eqs. (E.4). We now use induction and assume that

$$e_\ell = 1 \qquad \ell = M, M-1, \ldots, i$$

From Eqs. (E.12),

$$e_{i-1} = \frac{1}{p_{i,i-1}} \left[e_i - \sum_{j=i}^{M} p_{ij} \, e_j \right]$$

$$= \frac{1}{p_{i,i-1}} \left(1 - \sum_{j=i}^{M} p_{ij} \right) = 1$$

Thus, by induction we have shown that $e_i = f_i = 1$ for all i .

Using the preceding result, the solution to the set of M linear simultaneous equations in Eqs. (E.6) now becomes,

$$u_i = u_M + b_i \qquad i = 1, 2, \ldots, M - 1 \qquad \text{(E.18)}$$

and

$$u_M = -\frac{1}{p_{10}}\left(b_1 + 1 - \sum_{j=1}^{M-1} p_{1j}\, b_j\right) \qquad \text{(E.19)}$$

where we define

$$b_{M-1} = \frac{1}{p_{M,M-1}} \qquad \text{(E.20)}$$

and for $i = M - 1, M - 2, \ldots, 2$ we obtain recursively

$$b_{i-1} = \frac{1}{p_{i,i-1}}\left(b_i + 1 - \sum_{j=i}^{M-1} p_{ij}\, b_j\right) \qquad \text{(E.21)}$$

Similarly, the solution to the set of M linear simultaneous equations in Eqs. (E.7) becomes

$$w_i = w_M + d_i \qquad i = 1, 2, \ldots, M - 1 \qquad \text{(E.22)}$$

and

$$w_M = -\frac{1}{p_{10}}\left(d_1 - C_1 - \sum_{j=1}^{M-1} p_{1j}\, d_j\right) \qquad \text{(E.23)}$$

where we define

$$d_{M-1} = -\frac{C_M}{p_{M,M-1}} \qquad \text{(E.24)}$$

and for $i = M - 1, M - 2, \ldots, 2$ we obtain recursively

$$d_{i-1} = \frac{1}{p_{i,i-1}} \left(d_i - C_i - \sum_{j=i}^{M-1} p_{ij} \; d_j \right) \qquad \text{(E.25)}$$

Using Eqs. (E.5) to substitute for v_i in Eq. (E.1), we obtain

$$g = C_0 + \left(\sum_{j=1}^{M} p_{0j} \; u_j \right) g + \sum_{j=1}^{M} p_{0j} \; w_j$$

from which we get

$$g = \frac{C_0 + \sum_{j=1}^{M} p_{0j} \; w_j}{1 - \sum_{j=1}^{M} p_{0j} \; u_j} \qquad \text{(E.26)}$$

Finally, v_i are obtained using Eqs. (E.5). The derivation of Algorithm 6.5 is complete.

Some monotone properties

We show below monotone properties of the sequences b_i, d_i, u_i and w_i in Algorithm 6.5. The transition probabilities p_{ij} are assumed to satisfy Eqs. (E.4). The probabilities $p_{i,i-1}$ are assumed to be strictly positive. (This last is a necessary condition for the Markov process in Section 6.3 to be irreducible.)

Property E.1 The sequence b_i is positive and monotonically decreases to 0 as i increases to M.

Proof From Eqs. (E.10) and (E.11),

$$b_M = 0$$

$$b_{M-1} = \frac{1}{p_{M,M-1}} > b_M$$

The proof is by induction. Assume that b_ℓ decreases as ℓ increases for $i \le \ell \le M$. From Eqs. (E.12) and (E.4)

$$b_{i-1} = \frac{1}{p_{i,i-1}} \left[b_i + 1 - \sum_{j=i}^{M} p_{ij} b_j \right]$$

$$> \frac{1 + b_i p_{i,i-1}}{p_{i,i-1}} > b_i$$

Q.E.D.

Property E.2 (i) If C_i are positive, the sequence d_i is negative and monotonically increases to 0 as i increases to M. (ii) If C_i are negative, the sequence d_i is positive and monotonically decreases to 0 as i increases to M.

Proof The proof uses Eqs. (E.4), (E.15), (E.16) and (E.17), and is similar to that of Property E.1.

<u>Property E.3</u> The sequence u_i is negative and monotonically decreases as i increases.

<u>Proof</u> From Eq. (E.19)

$$u_M = -\frac{1}{p_{10}}\left(b_1 + 1 - \sum_{j=1}^{M-1} p_{1j}\ b_j\right)$$

$$-u_M = \frac{1}{p_{10}}\left(1 + b_1 - \sum_{j=1}^{M-1} p_{1j}\ b_j\right)$$

$$> \frac{1 + b_1\ p_{10}}{p_{10}} > b_1$$

where b_1 is positive from Property E.1. From Eq. (E.18)

$$u_j = u_N + b_j$$

Applying Property E.1, the proof is complete.

Q.E.D.

<u>Property E.4</u> (i) If C_i are positive, the sequence w_i is positive and monotonically increases as i increases. (ii) If C_i are negative, the sequence w_i is negative and monotonically decreases as i increases.

<u>Proof</u> The proof uses Eqs. (E.22) and (E.23) and Property E.2. The proof is similar to that of Property E.3.

APPENDIX F

A GENERAL DYNAMIC CHANNEL CONTROL PROCEDURE

In this appendix, a dynamic channel control procedure is formulated which includes as special cases ICP, RCP and IRCP in Chapter 6. Lemma 6.3 and Theorem 6.4 on the equivalence of the performance measures for ICP, RCP and IRCP are then extended to this general case.

Consider the action space $A_1 = \{\beta_1, \beta_2, \ldots, \beta_m\}$ where $0 \le \beta_1 < \beta_2 < \ldots < \beta_m \le 1$, and the action space $A_2 = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ where $0 < \gamma_1 < \gamma_2 < \ldots < \gamma_k < 1$. Let $A = A_1 \times A_2$ such that each element in A is a two-dimensional vector (β, γ). As in Section 6.3, the Markov decision process N^t has a finite state space $S = \{0, 1, 2, \ldots, M\}$. A stationary control policy f maps S into A. Given a stationary control policy f, $f(i) = (\beta, \gamma)$ means that whenever $N^t = i$, each (new) packet arrival is accepted with probability β (and rejected with probability $1 - \beta$) while each backlogged packet is retransmitted with probability γ in the t^{th} time slot. Thus, ICP corresponds to the special case $A = \{0, 1\} \times \{p_o\}$; RCP corresponds to the special case $A = \{1\} \times \{p_o, p_c\}$; IRCP corresponds to the special case $A = \{0, 1\} \times \{p_o, p_c\}$.

State Transition Probabilities

Suppose N^t is in state i and the stationary control policy $f(i) = (\beta, \gamma)$, then the one-step state transition probabilities are given by

$$
p_{ij}(f) = \begin{cases} 0 & j \le i - 2 \\ i\gamma(1-\gamma)^{i-1}(1-\beta\sigma)^{M-i} & j = i - 1 \\ (1-\gamma)^{i}(M-i)\beta\sigma(1-\beta\sigma)^{M-i-1} & \\ \qquad + \left[1 - i\gamma(1-\gamma)^{i-1}\right](1-\beta\sigma)^{M-i} & j = i \\ \left[1-(1-\gamma)^{i}\right](M-i)\beta\sigma(1-\beta\sigma)^{M-i-1} & j = i + 1 \\ \binom{M-i}{j-i}(\beta\sigma)^{j-i}(1-\beta\sigma)^{M-j} & j \ge i + 2 \\ & 0 \le i,\ j \le M \end{cases} \tag{F.1}
$$

Stationary Channel Throughput Rate

Suppose N^t is in state i and $f(i) = (\beta, \gamma)$. Define the expected immediate cost to be

$$
\begin{aligned} C_i(f) &= -S_{out}(i, f) \\ &= -\Big[i\gamma(1-\gamma)^{i-1}(1-\beta\sigma)^{M-i} \\ &\qquad + (1-\gamma)^{i}(M-i)\beta\sigma(1-\beta\sigma)^{M-i-1}\Big] \end{aligned} \tag{F.2}
$$

By Eq. (6.9) the cost rate of N^t is

$$
g_s(f) = -\sum_{i=0}^{M} \pi_i(f)\, S_{out}(i, f)
$$

Then, the stationary channel throughput rate is given by Eq. (6.30) which we rewrite below.

$$
S_{out} = -g_s(f) \tag{F.3}
$$

Average Packet Delay

Suppose N^t is in state i and $f(i) = (\beta, \gamma)$. Define the expected immediate cost to be

$$C_i(f) = i + (M - i)(1 - \beta)\sigma d_r \tag{F.4}$$

where d_r is the expected cost in units of delay per packet arrival rejected and is equal to $\frac{1}{\sigma}$ (see Section 6.3.3).

Let $S = \bigcup_{\ell=1}^{m} S_\ell$ where $S_1, S_2, \ldots, S_m$ are nonintersecting sets corresponding to a stationary control policy f such that

$$f(i) = (\beta_\ell, \gamma) \text{ if and only if } i \in S_\ell$$

where $\ell = 1, 2, \ldots, m$ and γ is any action in A_2.

By Eq. (6.9), the cost rate of N^t is

$$\begin{aligned} g_d(f) &= \sum_{i=0}^{M} C_i(f)\pi_i(f) \\ &= \sum_{i=0}^{M} i\ \pi_i(f) + \sum_{\ell=1}^{m} \sum_{i \in S_\ell} (M - i)(1 - \beta_\ell)\sigma d_r \pi_i(f) \\ &= \bar{N} + \lambda_r d_r \\ &= \bar{N} + \bar{N}_r \end{aligned} \tag{F.5}$$

where

$$\lambda_r = \sum_{\ell=1}^{m} \sum_{i \in S_\ell} (M - i)(1 - \beta_\ell)\sigma\ \pi_i(f) \tag{F.6}$$

is the stationary packet rejection rate; $\bar{N}$ is the average channel backlog size and $\bar{N}_r$ is the average number of rejected packets in the system.

Using Little's result [LITT 61] the average packet delay is given by Eq. (6.31) which we rewrite below.

$$D = \frac{g_d(f)}{S_{out}} + R + 1$$

$$= - \frac{g_d(f)}{g_s(f)} + R + 1 \qquad (F.7)$$

We give below an extension of Lemma 6.3 to the general dynamic channel control procedure.

Lemma F.1 Given any stationary control policy $f: \mathcal{S} \to \mathcal{A}$

$$g_d(f) = \frac{g_s(f)}{\sigma} + M$$

Proof From Eq. (F.5) and $d_r = \frac{1}{\sigma}$

$$g_d(f) = \sum_{i=0}^{M} i \; \pi_i(f) + \sum_{i=0}^{M} (M - i) \; \pi_i(f) - \frac{1}{\sigma} \sum_{\ell=1}^{m} \sum_{i \varepsilon S_\ell} (M - i) \; \beta_\ell \sigma \; \pi_i(f)$$

$$= M - \frac{1}{\sigma} \sum_{\ell=1}^{m} \sum_{i \varepsilon S_\ell} (M - i) \beta_\ell \; \sigma \; \pi_i(f)$$

Note that $\sum_{\ell=1}^{m} \sum_{i \varepsilon S_\ell} (M - i) \beta_\ell \; \sigma \; \pi_i(f)$ is just the stationary channel input rate and is thus equal to the stationary channel throughput rate $S_{out} = - g_s(f)$. Hence,

$$g_d(f) = \frac{g_s(f)}{\sigma} + M$$

Q.E.D.

With the above lemma, Theorem 6.4 can then be extended to the general dynamic channel control procedure.