Arithmetic Mean is greater than or equal to Geometric Mean

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I show that the arithmetic mean, a, is at least the geometric mean, g, for a finite bag B of non-negative real numbers. If B includes a 0, then $a \geq 0$ and $g = 0$. Henceforth, assume that B contains only positive reals.

Theorem 1: $a \geq g$. The following proof is adapted from a proof from *An Introduction to Inequalities*, section 11, by Beckenbach and Bellman, 1961.

The essential idea is "scaling", multiplying all elements of a bag by a positive real number. Scaling by r converts the arithmetic mean a and geometric mean g to ra and rg, respectively, and $a \geq g \equiv ra \geq rg$. Choose r to be the reciprocal of the product of the elements of the bag, so the product after scaling is 1 and $g = 1$. It is then sufficient to show that the arithmetic mean is at least 1. Equivalently, prove that for any non-empty bag B of n positive real numbers whose product is 1, the sum of its elements, $sum(B)$, is at least n. Proof is by induction on n.

For $n = 1$, the arithmetic and geometric mean are identical, so the result holds. For the inductive step let B have $n+1$ elements, $n \geq 1$. Choose distinct elements x and y from B where x is a smallest and y a largest element. Since the product of the elements is 1, $x \leq 1$, and, similarly, $y \geq 1$. Let B' be the bag obtained by replacing x and y in B by their product, i.e. $B' = B - \{x, y\} \cup \{x \times y\}$ so that the product remains equal to 1. Our goal is to prove that $sum(B) \geq n+1$.

$$
\begin{array}{cl}
 & sum(B) \\
= & \{\text{Definition of } B'\} \\
 & sum(B') + x + y - x \times y \\
\geq & \{x \leq 1,\ y \geq 1 \Rightarrow (1-x)(y-1) \geq 0, \text{ or } x + y - x \times y \geq 1\} \\
 & sum(B') + 1 \\
\geq & \{sum(B') \geq n, \text{ from the induction hypothesis}\} \\
 & n + 1
\end{array}
$$

Note Added; 9/28/06 The following proof of Theorem 1, which appears in Hardy, Littlewood and Polya, was shown to me by Anindya Patthak. First, prove the result for all bags whose sizes are powers of 2. Then show that if the result holds for all bags of size n, $n > 1$, it holds for all bags of size $n - 1$ as well.

• Proof for bags whose sizes are powers of 2: The result holds trivially for bags of size 2^0.

For a bag of size 2, say with elements x and y:

$$
\begin{array}{ll}
 & (x+y)^2 \\
= & \{\text{algebra}\} \\
 & (x-y)^2 + 4xy \\
\geq & \{(x-y)^2 \geq 0\} \\
 & 4xy
\end{array}
$$

It follows that $1/2(x+y) \geq \sqrt{xy}$.

For a bag of size 2^n, $n > 1$:

Divide the bag into two bags of equal sizes whose arithemetic means are a and a' and geometric means are g and g', respectively. Inductively, $a \geq g$ and $a' \geq g'$. The arithmetic mean of the original bag is $1/2(a+a')$ and the geometric mean is $\sqrt{gg'}$.

$$
\begin{array}{ll}
 & 1/2(a+a') \\
\geq & \{a \geq g \text{ and } a' \geq g'\} \\
 & 1/2(g+g') \\
\geq & \{\text{consider the bag } \{g, g'\}\text{; from the last proof}\} \\
 & \sqrt{gg'}
\end{array}
$$

• Given that the result holds for all bags of size n, it holds for all bags of size $n-1$:

Given is a bag B of size $n-1$. Let the sum of its elements be A, product be G, arithmetic mean be a and geometric mean be g. Therefore, $a = (1/(n-1))A$ and $g = G^{1/(n-1)}$. Consider the bag $B' = B \cup \{a\}$. The sum and product of B' are $A + a$ and Ga, respectively.

First, observe that the arithmetic mean of B' is a, because $(1/n)(A+a) = (1/n)(A + (1/(n-1))A) = (1/n)(n/(n-1))A = (1/(n-1))A = a$. And, the geometric mean of B' is $(Ga)^{1/n}$. From the inductive hypothesis, for B'

$$
\begin{array}{ll}
 & a \geq (Ga)^{1/n} \\
\Rightarrow & \{\text{algebra}\} \\
 & a^n \geq Ga \\
\Rightarrow & \{\text{algebra}\} \\
 & a^{n-1} \geq G \\
\Rightarrow & \{\text{algebra}\} \\
 & a \geq G^{1/(n-1)} \\
\Rightarrow & \{g = G^{1/(n-1)}\} \\
 & a \geq g
\end{array}
$$