1. (Compression; 19 points)
   (a) (4 points) Compute the entropy of the alphabet \([a, b, c, d]\) with associated probabilities \([\frac{1}{2}, \frac{1}{8}, \frac{1}{4}, \frac{1}{8}]\).
   (b) (7 points) Let \(T\) be the tree corresponding to an optimal prefix code. Show that every non-leaf node in \(T\) has two sons. You may assume that every symbol has non-zero probability of occurrence.
   (c) (8 points) Let \(R\) be a set of symbols which includes two specific symbols, \(x\) and \(y\). Let \(S = R - \{x, y\} \cup \{z\}\) where \(z\) is a new symbol not in \(R\). Use \(x\), \(y\) and \(z\) also for the probabilities associated with these symbols, and suppose \(z = x + y\). Let \(r\) and \(s\) be the weights of the optimal trees of \(R\) and \(S\), respectively. How are \(r\) and \(s\) related?

2. (Powerlist, 18 points)
   (a) (9 points) For each natural number \(i\), \(u_i\) and \(v_i\) are powerlists, defined as follows.
      \[
      u_0 = \langle 0 \rangle, \quad v_0 = \langle 1 \rangle \\
      \text{For all } i, i \geq 0:\n      u_{i+1} = u_i | v_i, \text{ and} \\
      v_{i+1} = v_i | u_i
      \]
      Show that for all \(i, i \geq 0:\n      u_{i+1} = u_i \triangledown v_i, \text{ and} \\
      v_{i+1} = v_i \triangledown u_i
      \]
      Hint: You may prove only one part, say, \(u_{i+1} = u_i \triangledown v_i\); the other proof is symmetric.
   (b) (9 points) Define the prefix function over powerlists. Write \(p \sqsubseteq q\) to denote that powerlist \(p\) is a prefix of powerlist \(q\). Use a pseudo programming notation, similar to the one used in the class for powerlists.

3. (String Matching; 13 points)
   (a) (Core computation; 6 points) You are given a string \(v[0..20]\) and told that \(v[6] \neq v[11]\). Which prefix of \(v\) can not be its core?
   (b) (Core computation; 7 points) Prove or disprove: for non-empty \(u\) and \(v\),
      \[
      u \preceq v \Rightarrow c(u) \preceq c(v)
      \]