CS 337 **Solution to Quiz 3** **5/8/03**

1. (Compression)

   (a) entropy

   $= -1/2\log(1/2) - 1/8\log(1/8) - 1/4\log(1/4) - 1/8\log(1/8)$
   $= -1/2(-1) - 1/8(-3) - 1/4(-2) - 1/8(-3)$
   $= 1/2 + 3/8 + 2/4 + 3/8$
   $= 1.75$

   (b) Let T have a non-leaf node x whose only son is y; we show that T is not optimal by displaying another tree T', which is also a prefix code, whose weight is strictly lower.

   If x is the root of T create T' by deleting x and letting y be the root. If x is non-root, change the parent of y to the parent of x, and delete x. In each case, the pathlengths to the descendants of y decrease and other pathlengths do not increase. So, the weight of T' is less than that of T.

   (c) Let the weight of the optimal trees for S and R be s and r, respectively. We can get a tree for R from the one for S by expanding the leaf z into a non-leaf with children x and y. The weight of the resulting tree is $s + x + y$. This quantity is at least the weight of the optimal tree for R, r. That is,

   $$r \le s + x + y$$

2. (Powerlist)

   (a) Proof is by induction on i.

   - $i = 0$: We have to show $u_1 = u_0 \bowtie v_0$ and $v_1 = v_0 \bowtie u_0$.

   $$\begin{array}{ll} & u_1 \\ = & \{\text{definition of } u_1\} \\ & u_0 \mid v_0 \\ = & \{\text{laws of powerlist: } u_0 \text{ and } v_0 \text{ are singletons}\} \\ & u_0 \bowtie v_0 \end{array}$$

   - $i > 0$:

   $$\begin{array}{ll} & u_{i+1} \\ = & \{\text{definition of } u_{i+1}\} \\ & u_i \mid v_i \\ = & \{\text{induction on both terms}\} \\ & (u_{i-1} \bowtie v_{i-1}) \mid (v_{i-1} \bowtie u_{i-1}) \\ = & \{\text{commutativity}\} \\ & (u_{i-1} \mid v_{i-1}) \bowtie (v_{i-1} \mid u_{i-1}) \\ = & \{\text{definitions of } u_i \text{ and } v_i\} \\ & u_i \bowtie v_i \end{array}$$

(b) $$p \sqsubseteq \langle x \rangle = (p == \langle x \rangle)$$
$$p \sqsubseteq r \mid s = (p == r \mid s) \vee (p \sqsubseteq r)$$

Another possible definition is

$$\langle x \rangle \sqsubseteq \langle y \rangle = (x == y)$$
$$\langle x \rangle \sqsubseteq r \bowtie s = (\langle x \rangle \sqsubseteq r)$$
$$p \bowtie q \sqsubseteq r \bowtie s = (p \sqsubseteq r) \wedge (q \sqsubseteq s)$$

3. (String Matching)

(a) Suppose $v[0..k]$ is the core. From the definition of core,

$v[0..k] = v[20-k..20]$. Hence,
$v[i] = v[20-k+i]$

Letting $i = 6$ and $20 - k + i = 11$, we get $k = 15$. That is, if $k = 15$, $v[6] = v[11]$. Since $v[6] \neq v[11]$, $v[0..15]$ is not the core.

(b)
$$
\begin{array}{ll}
 & u \preceq v \\
\Rightarrow & \{c(u) \prec u\} \\
 & c(u) \prec v \\
\Rightarrow & \{\text{definition of core: } w \prec v \ \equiv \ w \preceq c(v)\} \\
 & c(u) \preceq c(v)
\end{array}
$$