An elementary proof of Hall's marriage theorem

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1 Introduction

Hall's marriage theorem [1] is applied in many combinatorial problems. Given is a bipartite graph B with non-empty node sets X and Y. A *matching* is a set of edges that have no common incident nodes. A (X, Y) matching is a matching in which every node of X is incident on some edge in the matching.

Hall condition (HC): Subset S of X *meets* HC if the number of neighbors of S is greater than or equal to the size of S.

Theorem 1 [Hall] There is a (X, Y) matching if and only if every subset of X meets HC.

Proof: The proof in one direction, that if there is a (X, Y) matching every subset of X meets HC, is straightforward. I prove the converse of the statement by induction on the size of set X. If X is empty, there is a trivial matching. For the general case assume, using induction, that there is a matching over all nodes of X except one node r.

Henceforth $u \overset{n}{—} v$ and $u \overset{m}{—} v$ denote, respectively, that (u, v) is a non-matching edge and (u, v) a matching edge. An *alternating path* is a simple path of alternating matching and non-matching edges. Let Z be the subset of nodes of X that are connected to r by an alternating path.

Every node of Z except r is connected to a unique node in Z' by a matching edge, from the induction hypothesis; so, there are $|Z| - 1$ nodes in Z' that are so connected. Since Z meets HC, $|Z'| \geq |Z|$. Therefore, there is a node v in Z' that is not connected to any node in Z by a matching edge. I show that v is not incident on any matching edge.

Let v be a neighbor of u in Z, so $u \overset{n}{—} v$. Since $u \in Z$, there is an alternating path between r and u; extend the path to include edge $u \overset{n}{—} v$, as shown below in P. I color the matching edges blue and non-matching edges red for emphasis.

$P: \; r = x_0 \overset{n}{—} y_0 \overset{m}{—} x_1 \cdots x_i \overset{n}{—} y_i \overset{m}{—} x_{i+1} \cdots x_t \overset{n}{—} y_t \overset{m}{—} x_{t+1} = u \overset{n}{—} v.$

If v is incident on a matching edge, say $v \overset{m}{-} w$, then $w \notin Z$ because v is not connected to any node in Z by a matching edge. However, we can extend P as follows which shows that r is connected to w by an alternating path, so $w \in Z$, a contradiction.

P' : $r = x_0 \overset{n}{-} y_0 \overset{m}{-} x_1 \cdots x_i \overset{n}{-} y_i \overset{m}{-} x_{i+1} \cdots x_t \overset{n}{-} y_t \overset{m}{-} x_{t+1} = u \overset{n}{-} v \overset{m}{-} w.$

So, we conclude that v is not incident on any matching edge.

Flip the edge labels in P from n to m and m to n to obtain a matching that includes all previously matched nodes of X and now includes r, so all nodes of X are in the matching.

$r = x_0 \overset{m}{-} y_0 \overset{n}{-} x_1 \cdots x_i \overset{m}{-} y_i \overset{n}{-} x_{i+1} \cdots x_t \overset{m}{-} y_t \overset{n}{-} x_{t+1} = u \overset{m}{-} v.$

This completes the inductive proof.

References

[1] Philip Hall. On representatives of subsets. *J. London Math. Soc.*, 10(1):26–30, 1935.