A simple and neat denotational semantic theory of concurrent systems

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In Honor of Jose Meseguer
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A Quote from H. L. Mencken, American Essayist, 1930s

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Research Connections with Jose Meseguer

- Thesis of Mark-Oliver Stehr
  Includes extending and explaining the Unity logic

- Thesis of Musab AlTurki
  Includes extending and explaining the Orc real-time semantics

- Orc can be subsumed within Maude, very easily

- And much much more.

Motivation for the current work: Commutative, Associative Fold

- Bag u.

  Commutative, associative binary operator $\oplus$

  Write fold of u as Σu.

- Problem: Replace all elements of u by Σu.

- Strategy: Define f_k:
  - reduces u by k in size, and
  - the resulting bag has the same fold as the original bag.

An Orc Program

$$f_1 = get(x);\ get(y);\ put(x \oplus y)$$

$$f_{k+1} = f_1 \ [\!]\ f_k, \quad k \geq 1$$

Apply $f_{|u_0|-1}$.

- No known proof technique for this program.
- I attempted using denotational semantics.
- Wrote a paper. Mailed to Jose.

Response from Jose

I have read carefully your very interesting paper draft over the last three days, have hand-written many detailed comments on the draft, and written also a good number of additional pages with further comments. I am traveling today by train to Madrid and will fly back to Urbana tomorrow.

There are some quite interesting and I think useful connections with some category theory results on completion of posets under various kinds of limits that I worked on in the 1980s that I would like to have the chance to relate in more detail to your constructions;

(3) There is a ~~k~~ 1-to-1 correspondence because of ⊛ in page 16 between:

(a) chain-continuous "transformers"

$$f : (A, \leq) \longrightarrow (\tilde{A}, \subseteq)$$

and

(b) chain-continuous $\tilde{f} : (\tilde{A}, \subseteq) \to (\tilde{A}, \subseteq)$

such that $f = \tilde{f} \circ j_A$

So, the moral of the story is that to get the appropriate continuity for limits of chains we should focus not on $Id_\beta^{Filter}(A, \leq)$ (the upward-closed ~~sp~~ smooth specs) but on its subposet $(\tilde{A}, \subseteq) \subseteq (Id_\beta^{Filter}(A, \leq), \subseteq)$ which is the completion by limits of chains of $(A, \leq)$ ~~that preserves~~ such that j_A ~~is~~ preserves

My email afterwards

Jose: There is just one way to describe your comments on my manuscript: awesome. It is awesome because I can not imagine replicating something of this nature myself for someone else

...

I am eternally grateful to you, not just for your comments, but for being a friend.

Disgusting Anticlimax

- Could not prove the fold program.
- But got many interesting insights about concurrency, semantic theory and my overall ignorance in these areas.

Denotational Semantics of Concurrent Systems

- Scott's denotational semantics specialized to concurrent systems.
- Strong results for this specific domain.
- Inappropriate for other areas, such as sequential programs.
- Derive specification of a program from those of its components.

Denotational Semantics

- $f \oplus g$ is a program constructed out of

  components f and g, and

  combinator $\oplus$, a programming language construct.

- Specifications of f and g appear as $[\![f]\!]$ and $[\![g]\!]$.

- The specification of $f \oplus g$, $[\![f \oplus g]\!]$, is given by:

  $$[\![f \oplus g]\!] \triangleq [\![f]\!] \; [\![\oplus]\!] \; [\![g]\!]$$

- $[\![\oplus]\!]$ is a transformer of specifications:

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Notation Overloading: use $\oplus$ instead of $[\![\oplus]\!]$.

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Contributions of this work

- Specifications of components.
- A theory of transformers: functions mapping specs to specs.
- Treated:

  concurrency

  non-determinacy

  recursion

  shared resource

  fairness

  divergence

  real-time

Summary

Closure	Meaning	Preserving Transformer	Corresponding Function
Downward	Prefix-closed	Smooth	Monotonic
Upward	Limit-closed	Bismooth	Continuous

- A library of smooth and bismooth transformers.
- Fixed-point theorems:
  - Least upward-closed fixed point
  - Min-max fixed point (to deal with fairness)

Component Specification

- Events.
- Traces.
- A specification is a prefix-closed set of traces.

Events associated with a component

$pub(true)$	publish (output) a value
$x.read(3)$	read value 3 from variable x
$c.receive(''val'')$	receive $''val''$ from channel c
$Heads/Tails$	outcome of a coin toss
$x.add(5)$	Method call

- Events are event instances.
- They are uninterpreted, instantaneous and atomic.
- There is a universal event alphabet.

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Execution of a component (informal notion)

An execution is a sequence of events.

Toss a coin and publish the outcome.
Two possible executions:

$$[Heads, pub(\text{"}Heads\text{"})]$$
$$[Tails, pub(\text{"}Tails\text{"})]$$

With all intermediate executions:

$$[\,]$$
$$[Heads]$$
$$[Heads, pub(\text{"}Heads\text{"})]$$
$$[Tails]$$
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$$[Tails]$$
$$[Tails, pub("Tails")]$$

Another Program

Two tosses, but stop if the first toss is Heads

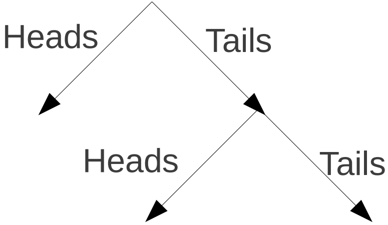
$[Heads]$
$[Tails, Heads]$
$[Tails, Tails]$

Plus all the prefixes of these sequences.

Depict Executions by a tree

Two tosses, but stop if the first toss is Heads

$[Heads]$, $[Tails, Heads]$, $[Tails, Tails]$ plus the prefixes.



- Each node is an execution.
- Label on each branch is an event.
- An ancestor is a prefix.

Infinite Executions

Toss a coin repeatedly until it lands Heads.

$$[\,]$$
$$[Heads] \qquad [Tails]$$
$$[Tails, Heads] \qquad [Tails, Tails]$$
$$[Tails, Tails, Heads] \qquad [Tails, Tails, Tails]$$
$$[Tails, Tails, Tails, Heads] \cdots$$

- An unfair coin may may always land Tails.
- Admit infinite execution: $[Tails, Tails, Tails, \cdots]$
- Executions described by:

$$\{[Tails^j] \mid j \geq 0\} \cup \{[Tails^j, Heads] \mid j \geq 0\} \cup \{[Tails^\omega]\}$$

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Status of an Execution

- Status denotes the final state of an execution. From $\{W, H, D\}$.

- Infinite execution has status D.

- Finite executions typically have status H or W. Some have D.

  W is Waiting:
  more autonomous computation to do or waiting for external input.

  H is Halted: nothing more to do.

  D is Divergent: An infinite computation.

- Example of Divergent Execution

```
def loop( ) = loop( )
```

Trace

A trace is $s[m]$ where

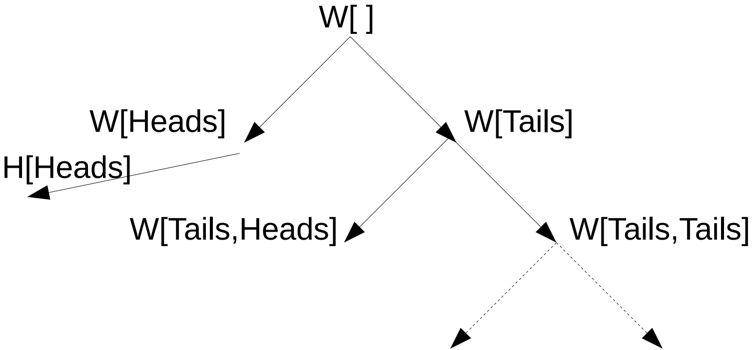
- s, status, is from $\{W, H, D\}$.
- m finite or infinite event sequence.

Trace (formal notion)

Trace: A sequence of events plus the final state of computation.

Toss a coin repeatedly until it lands Heads:

$W[\,]$	$W[Heads]$	$W[Tails]$	
	$H[Heads]$	$W[Tails, Heads]$	$W[Tails, Tails]$



Trace prefix

In the trace tree, prefix of a node is an ancestor.

Formally, $s[m] \leq s'[m']$, means

$s[m] = s'[m']$, or

$(s = W)$ and $(m$ prefix of $m')$

Applies to infinite traces.

- $\leq$ is a partial order.
- $>$ is a well-founded order.
- $W[\,]$ is the bottom trace.

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Prefix Closure (downward closure)

Prefix closure of trace t is the set of all its prefixes:

$$t_* = \{s \mid s \leq t\}$$

For traceset (non-empty set of traces) p define downward closure by:

$p_* = \cup_{t \in p}(t_*)$, for non-empty p

$(p \times q \times \cdots \times r)_* = p_* \times q_* \cdots \times r_*$ Cartesian Product

Spec

- A specification (spec) is a non-empty prefix-closed set of traces, i.e., $p = p_*$.

Meaning of spec

- Each trace in a spec of f is a possible execution of f in some environment.
- So, a spec is prefix-closed.
- Deadlock: A spec that includes $W[m]$ but no extension.
- Eventual halting:
  - Every waiting trace has an extension by an autonomous event.
  - There is no divergent trace.

Tree depiction of a spec is insufficient

Toss a coin sequentially until it lands Heads.

unfair coin: $\{H[Tails^j, Heads] \mid j \geq 0\}_* \cup \{D[Tails^\omega]\}$

fair coin: $\{H[Tails^j, Heads] \mid j \geq 0\}_*$

Explicit inclusion/exclusion of infinite traces in a spec.

Denotational Semantics (repeated)

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  Notation Overloading: use $\oplus$ instead of $[\![\oplus]\!]$.

A Motivating Example

- Programming language construct, $\oplus$: $\oplus(A, B, C)$
- Execute A, B concurrently.
- If A engages in e and B in $\overline{e}$, they rendezvous. Then start C to run concurrently with A and B.

A Motivating Example: $\oplus(A, B, C)$

- Let specifications of A, B, C be p, q, r, respectively.
- C' starts with event a and then behaves as C:
  spec is $cons(a, r)$.
- spec of A, B, C' running concurrently: $p \mid q \mid cons(a, r)$.
- Retain those traces in which $\{e, \overline{e}, a\}$ are contiguous.
  Replace these 3 events by event τ:
  $rendezvous(\{e, \overline{e}, a\}, \tau, (p \mid q \mid cons(a, r)))$
- Drop the τ symbol from each trace:
  $\oplus'(p, q, r) = drop(\tau, rendezvous(\{e, \overline{e}, a\}, \tau, (p \mid q \mid cons(a, r))))$

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Example Transformer: Sequential Composition, $f ; g$

- g starts executing when and only when f halts.
- A trace of $f ; g$ is of the form:
  - $s[m]$ where $s[m]$ is a trace of f and s is W or D, or
  - $s[m\ n]$ where

    $H[m]$ is a trace of f

    $s[n]$ is a trace of g

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Example Transformer: parallel composition, $f \mid g$

- f and g execute independently.
- Let $s[m]$ be a trace of f, $t[n]$ of g, s and t from $\{H, W\}$.

  Then, $f \mid g$ includes traces $(s \cap t)(m \otimes n)$ where:

  - $\cap$ symmetric. $H \cap s = s$, $W \cap W = W$.
  - $m \otimes n$ is all interleavings (merge) of m and n.
- Merging with infinite sequence: fair and unfair merge.

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Definition: Transformer, Trace-wise Transformer

- A transformer is a function that maps a tuple of specs to a spec: $f(p, q, \cdots, r)$

  Notation: Infix $p \oplus q$ for 2-tuple transformer .

- Tracewise-transformer: Maps a tuple of traces to a traceset. Then,

  $$f(p) = \cup\{f(t) \mid t \in p\}$$

  $$p \oplus q = \cup\{s \oplus t \mid s \in p,\ t \in q\}$$

- Henceforth all transformers are trace-wise.

When is $f(p)$ a spec given that p is a spec?

Smooth Transformer

- A smooth transformer preserves prefix closure.
- Smooth Transformer: For any trace s,

$$f_*(s) = f(s_*) \qquad \text{(Notation: } f_*(s) \text{ is } (f(s))_*\text{)}$$
$$(s \oplus t)_* = s_* \oplus t_*$$

Properties of smooth transformers

- For smooth f and spec p, $f_*(p) = f(p_*)$.
- Follows: A smooth transformer transforms specs to specs.
- Composition of smooth transformers is smooth.
- f is smooth iff
  - f transforms specs to specs, and
  - f is monotonic: $s \leq t \Rightarrow f_*(s) \subseteq f_*(t)$.

Example of Smooth Transformer: choice

- f *or* g: choose to execute either f or g

  transformer: $s \; or \; t = \{s\} \cup \{t\}$

- *or* is smooth.

Example of Smooth Transformer: cons

- Append a specific event a as the first event of every trace.
- $$cons(a, W[\,]) = \{W[\,], W[a]\}$$
  $$cons(a, s[m]) = \{s[a\ m]\}$$

Example of Smooth Transformer: Filter

- A filter transformer accepts or rejects each trace.
- A *filter* is defined by a predicate b on traces, where
  1. $b(W[\,])$ holds, and
  2. If $b(t)$ holds then $b(s)$ holds for all prefixes s of t.
- A filter transformer accepts all prefixes for which b holds.

$$f(t) = \{s \mid b(s) \wedge s \leq t\}$$

Examples of Smooth transformers

- unfair merge: $f \mid g$
- fair merge: $f \mid' g$
- rendezvous: merge traces so that events e and e' are contiguous.
- sequential composition: $f \; ; g$

  $H[m] \; ; t[n] = \{t[m\ n]\}$,

  $s \; ; t[n] = \{s\}$, otherwise

Fairness

- Coin tosses are fair.
- Fair scheduler: In a multiprocess implementation every process gets to execute eventually.
- A semaphore is granted fairly.
- Any finite interval in time can contain only a finite number of events.

Fairness is a filter transformer

- The transformer accepts all finite traces,
  accepts the fair infinite traces and rejects the unfair ones.
- Fits the definition of a filter, a smooth transformer.

Example: coin toss forever until Heads appears.

- unfair coin:

$$\{H[Tails^j, Heads] \mid j \geq 0\}_* \cup \{D[Tails^\omega]\}$$

- fair coin: Apply the filter that rejects the infinite sequence of Tails.

$$\{H[Tails^j, Heads] \mid j \geq 0\}_*$$

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Shared Resource

- Consider $x.read() \mid x.write(3)$,
  where local variable x is initialized to 0.

- spec of $x.read()$ includes the trace $H[read(5)]$.

  spec of program $x.write(3)$ is $H[write(3)]_*$

- Applying merge: a trace of $x.read() \mid x.write(3)$ is

  $H[read(5), write(3)]$, an invalid trace.

Parallel executions may not be independent

- The complete program is

  int $x = 0$

  $x.read() \mid x.write(3)$

- The declaration "int $x = 0$" induces a filter transformer, $x.int$.

  It rejects all traces that are not possible with the resource.

- Given specs p and q of $x.read()$ and $x.write(3)$, spec of

  int $x = 0$

  $x.read() \mid x.write(3)$

  is $x.int(p \mid q)$

Research Area

- Each shared resource is defined by a filter.
- Each filter is an acceptor of strings, i.e., a formal language.
- So, a shared resource can be specified as a language.
- The language may include infinite strings, say, for strong semaphore.
- I have defined filters for
  read/write shared variables,
  write-once variables,
  channel,
  weak and strong semaphore

Recursion: Procedure $stut()$

- Toss an unfair coin
  if it lands Heads halt, otherwise call $stut()$.

- Let the spec of $stut()$ be x.

- $stut()$ chooses between
  - halting the computation (when toss lands Heads), with spec $H[\,]$, and
  - event Tails followed by $stut()$, with spec $cons(Tails, x)$
  - The transformer for choice is set union.

- $x = H[\,] \cup cons(Tails, x)$,
  $\cup$ and $cons$ are smooth.

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Solutions of recursive equation: $x = f(x)$

- Extensively studied in denotational semantics where x, called a point, is from a complete partial order (CPO).
  - There is a partial order $\subseteq$ in the cpo.
  - There is a bottom element, $\perp$.
  - Every chain $x_0 \leq x_1 \ldots$ has a least upper bound (lub) y:

    $x_i \subseteq y$ upper bound

    $y \subseteq z$ for any upper bound z.

- A solution of $x = f(x)$ is a fixed point of f.

  Wanted: the least fixed point, $lfp(f)$, according to $\subseteq$.

Least Fixed-point Theorem

- F is *continuous* means:
  For every chain C, $f(lub(C)) = lub(f(C))$.

- Theorem: Given $x = f(x)$ where f is *continuous*:

  $$lfp(f) = lub(f^i(W[\,]))$$

- That is, with

  $$x_0 = \bot,\ x_{i+1} = f(x_i),$$

  $$lfp(f) = lub(x_0, \ldots, x_i, \ldots)$$

In the current work

Specs form a complete partial order, where

- the order relation is subset order over specs, lub is set union,
- $\bot$ is the $W[\,]$,
- f, a smooth transformer is always continuous.
- Proposition: $lfp(f)$ is the expected outcome in an execution.

Example: $stut()$

- Recursive equation: $x = H[\,] \cup cons(Tails, x)$
- $lfp(stut) = \{H[Tails^j] \mid j \geq 0\}_*$
- This is not the correct solution.
  Does not include the infinite trace $D[Tails^\omega]$.

The fixed point should include the limit of all trace chains.

The crux of the problem

- We have ordered arbitrary specs by subset ordering.
  For a chain of specs $p_0 \subseteq p_1 \ldots$, lub is the union of the p_is.

- Consider only upward-closed specs. For a chain of such specs, the lub is upward-closure of their union.

Upward Closure

- Given trace chain C, $C = t_0 \leq t_1 \ldots$.
  Limit of C, $lim(C)$, the shortest trace that has every t_i as a prefix.

- Define upward closure of spec p as
  $p^* = p \cup \{lim(C) \mid C \text{ a chain in } p\}$

- Follows: for specs, $(p \times q \cdots \times r)^* = p^* \times q^* \cdots \times r^*$

least upward-closed fixed point (*lufp*)

- For recursive equation $x = f(x)$,
  the least upward-closed fixed point p is a spec such that:

$p = f(p)$	fixed point
$p = p^*$	upward-closed
$p \subseteq q$	for any upward-closed fixed point q

  Note: p is a spec, so downward-closed.

- $lufp(f)$ may not exist for arbitrary smooth f.

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Bismooth Transformer

- Smooth: $f(p_*) = f_*(p)$, for any traceset p
- Bismooth:

  Smooth (preserve downward-closure)

  Spec p: $f(p^*) = f^*(p)$ (preserve upward-closure)

Fairness is smooth but not bismooth.

Unfair merge is bismooth, fair merge only smooth.

Continuous filter is bismooth, discontinuous filter only smooth.

All other transformers seen so far are bismooth.

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Proving Bismoothness

- A transformer f maps a spec p, a tree of traces, to $f(p)$, another tree of traces.
- f smooth: maps every finite path x of p to a set of paths in $f(p)$.
- f bismooth: for every path y in $f(p)$ there is path x in p mapping to y.
- Use Koenig's infinity lemma: if p has finite degree, i.e., bounded non-determinism.

A variation of Koenig's infinity lemma

- S and T rooted trees. S domain spec, T range.
- $cover$: a binary relation over $S \times T$.
  Corresponds to a transformer from S to T.
- Node x of S covers node y of T means $(x, y) \in cover$.
  Also, y covered by x.
- Nodeset X covers Y (Y covered by X):
  every node of Y covered by some node of X.

Variation, Contd.

Theorem: Given S, T and cover as above, suppose:

- Each node of T is covered by a non-empty finite set of nodes of S.
- If node x covers node y then the ancestors of x in S (that includes x) cover the ancestors of y.

Then every path of T is covered by some path of S.

Sufficient Condition for Bismoothness

A transformer is co-finite means:

it maps a *finite* number of finite traces to any finite trace.

Theorem: A transformer that is smooth, co-finite and chain continuous is bismooth.

Least Upward-closed Fixed-point of Bismooth Transformer

Theorem: For bismooth f, $lufp(f) = lfp^*(f)$

Revisit $stut()$

- Recursive equation: $x = H[\,] \cup cons(Tails, x)$

- $lfp(stut) = \{H[Tails^j] \mid j \geq 0\}_*$

- $$\begin{aligned}
& lufp(stut()) \\
= \quad & \quad \{\text{From theorem}\} \\
& lfp^*(stut()) \\
= \quad & \quad \{\ lfp(stut) = \{H[Tails^j] \mid j \geq 0\}_* \} \\
& (\{H[Tails^j] \mid j \geq 0\}_*)^* \\
= \quad & \quad \{\text{computing}\} \\
& \{H[Tails^j \mid j \geq 0\}_* \cup \{D[Tails^\omega]\}
\end{aligned}$$

Fairness and Recursion

- Let $x = f(x)$ where f is smooth, not bismooth.
- f may have no upward-closed fixed point.
- maximal fixed-point: one that includes as many limit traces as possible (under the fairness constraint).
- the min-max fixed-point, $mmfp(f)$: the least maximal fixed-point.

Theorem: $mmfp(f)$ = the greatest fixed point of f in $lfp^*(f)$.