A simple and neat denotational semantic theory of concurrent systems

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In Honor of Jose Meseguer
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A Quote from H. L. Mencken, American Essayist, 1930s

For every complex problem there is a solution that is simple, neat and wrong.
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Research Connections with Jose Meseguer

- Thesis of Mark-Oliver Stehr
  Includes extending and explaining the Unity logic

- Thesis of Musab AlTurki
  Includes extending and explaining the Orc real-time semantics

- Orc can be subsumed within Maude, very easily

- And much much more.
Motivation for the current work: Commutative, Associative Fold

- Bag \( u \).
  Commutative, associative binary operator \( \oplus \)
  Write fold of \( u \) as \( \Sigma u \).

- Problem: Replace all elements of \( u \) by \( \Sigma u \).

- Strategy: Define \( f_k \):
  - reduces \( u \) by \( k \) in size, and
  - the resulting bag has the same fold as the original bag.
An Orc Program

\[ f_1 = \text{get}(x); \text{get}(y); \text{put}(x \oplus y) \]

\[ f_{k+1} = f_1 \parallel f_k, \quad k \geq 1 \]

Apply \( f_{|u_0| - 1} \).

- No known proof technique for this program.
- I attempted using denotational semantics.
- Wrote a paper. Mailed to Jose.
Response from Jose

I have read carefully your very interesting paper draft over the last three days, have hand-written many detailed comments on the draft, and written also a good number of additional pages with further comments. I am traveling today by train to Madrid and will fly back to Urbana tomorrow.

There are some quite interesting and I think useful connections with some category theory results on completion of posets under various kinds of limits that I worked on in the 1980s that I would like to have the chance to relate in more detail to your constructions;
There is a 1-to-1 correspondence between:

\[
\begin{align*}
\text{(a)} & \quad \text{chain-continuous } \quad \text{transform} \\
& \quad f: (A, \preceq) \rightarrow (\tilde{A}, \preceq)
\end{align*}
\]

and

\[
\begin{align*}
\text{(b)} & \quad \text{chain-continuous } \quad \tilde{f}: (\tilde{A}, \preceq) \rightarrow (\tilde{A}, \preceq) \\
& \quad \text{such that } \quad \tilde{f} = \tilde{f} \circ j_A
\end{align*}
\]

So, the moral of the story is that to get the approximate continuity in limits of chains we should focus on \( \text{Id}_\mathcal{P}(A, \preceq) \) (the upward-closed \& smooth map) but on its sub-predet \((\tilde{A}, \preceq) \in (\text{Id}_\mathcal{P}(A, \preceq), \preceq)\) which is the completion by limits of chains of \((A, \preceq)\) that \(j_A\) preserves.
Jose: There is just one way to describe your comments on my manuscript: awesome. It is awesome because I can not imagine replicating something of this nature myself for someone else ... 

I am eternally grateful to you, not just for your comments, but for being a friend.
Disgusting Anticlimax

- Could not prove the fold program.

- But got many interesting insights about concurrency, semantic theory and my overall ignorance in these areas.
Denotational Semantics of Concurrent Systems

- Scott’s denotational semantics specialized to concurrent systems.
- Strong results for this specific domain.
- Inappropriate for other areas, such as sequential programs.
- Derive specification of a program from those of its components.
Denotational Semantics

• $f \oplus g$ is a program constructed out of components $f$ and $g$, and combinator $\oplus$, a programming language construct.

• Specifications of $f$ and $g$ appear as $[f]$ and $[g]$.

• The specification of $f \oplus g$, $[f \oplus g]$, is given by:

$$[f \oplus g] \triangle [f] \oplus [g]$$

• $[\oplus]$ is a transformer of specifications:

It combines two specifications, $[f]$ and $[g]$, to yield a specification.

Notation Overloading: use $\oplus$ instead of $[\oplus]$. 
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[f \oplus g] \triangleq [f] \Box [g]
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  [f \oplus g] \Delta [f] [\oplus] [g]
  \]

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Notation Overloading: use \( \oplus \) instead of \([\oplus]\).
Contributions of this work

- Specifications of components.
- A theory of transformers: functions mapping specs to specs.
- Treated:
  - concurrency
  - non-determinacy
  - recursion
  - shared resource
  - fairness
  - divergence
  - real-time
## Summary

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- A library of smooth and bismooth transformers.
- Fixed-point theorems:
  - Least upward-closed fixed point
  - Min-max fixed point (to deal with fairness)
Component Specification

- Events.
- Traces.
- A specification is a prefix-closed set of traces.
Events associated with a component

\begin{itemize}
  \item \texttt{pub(true)} \quad publish (output) a value
  \item \texttt{x.read(3)} \quad read value 3 from variable \texttt{x}
  \item \texttt{c.receive("val")} \quad receive "val" from channel \texttt{c}
  \item \texttt{Heads/Tails} \quad outcome of a coin toss
  \item \texttt{x.add(5)} \quad Method call
\end{itemize}

- Events are event instances.
- They are uninterpreted, instantaneous and atomic.
- There is a universal event alphabet.
Events associated with a component

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- \textit{x.read(3)} reads value 3 from variable \textit{x}
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- Heads/Tails is the outcome of a coin toss
- \textit{x.add(5)} is a method call

- Events are event instances.
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- There is a universal event alphabet.
Execution of a component (informal notion)

An execution is a sequence of events.

Toss a coin and publish the outcome.
Two possible executions:

\[\text{[Heads, pub("Heads")]}\]
\[\text{[Tails, pub("Tails")]}\]

With all intermediate executions:

\[
\begin{align*}
&[] \\
&[\text{Heads}] \\
&[\text{Heads, pub("Heads")}] \\
&[\text{Tails}] \\
&[\text{Tails, pub("Tails")}] \\
\end{align*}
\]
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)] \\
&[\text{Tails}] \\
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\end{align*}
\]
Another Program

Two tosses, but stop if the first toss is Heads

[Heads]
[Tails, Heads]
[Tails, Tails]

Plus all the prefixes of these sequences.
Depict Executions by a tree

Two tosses, but stop if the first toss is Heads

\[\text{Heads}, \ [\text{Tails}, \text{Heads}], \ [\text{Tails}, \text{Tails}]\] plus the prefixes.

- Each node is an execution.
- Label on each branch is an event.
- An ancestor is a prefix.
**Infinite Executions**

Toss a coin repeatedly until it lands Heads.

\[
\begin{align*}
&\{ &\} \\
&[Heads] & [Tails] \\
&Tails, Heads & Tails, Tails \\
&Tails, Tails, Heads & Tails, Tails, Tails \\
&Tails, Tails, Tails, Heads & \cdots \\
\end{align*}
\]

- An unfair coin may always land Tails.

- Admit infinite execution: \([Tails, Tails, Tails, \cdots]\)

- Executions described by:

\[
\{[Tails^j] \mid j \geq 0\} \cup \{[Tails^j, Heads] \mid j \geq 0\} \cup \{[Tails^\omega]\}
\]
Infinite Executions

Toss a coin repeatedly until it lands Heads.

[ ]
[Heads] [Tails]
[Tails, Heads] [Tails, Tails]
[Tails, Tails, Heads] [Tails, Tails, Tails]
[Tails, Tails, Tails, Heads] ⋯

• An unfair coin may always land Tails.

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• Executions described by:

\{ [Tails]^j \mid j \geq 0 \} \cup \{ [Tails]^j, Heads \mid j \geq 0 \} \cup \{ [Tails^\omega] \}
Status of an Execution

• Status denotes the final state of an execution. From \{W, H, D\}.

• Infinite execution has status \( D \).

• Finite executions typically have status \( H \) or \( W \). Some have \( D \).
  
  \( W \) is Waiting: more autonomous computation to do or waiting for external input.
  
  \( H \) is Halted: nothing more to do.
  
  \( D \) is Divergent: An infinite computation.

• Example of Divergent Execution

  \[
  \text{def} \; \text{loop}(\; ) = \text{loop}(\; )
  \]
Trace

A trace is $s[m]$ where

- $s$, status, is from $\{W, H, D\}$.

- $m$ finite or infinite event sequence.
Trace (formal notion)

Trace: A sequence of events plus the final state of computation.

Toss a coin repeatedly until it lands Heads:

\[ W[\ ] \quad W[Heads] \quad W[Tails] \]
\[ H[Heads] \quad W[Tails, Heads] \quad W[Tails, Tails] \]
Trace prefix

In the trace tree, prefix of a node is an ancestor.

Formally, \( s[m] \leq s'[m'] \), means

\[
    s[m] = s'[m'], \text{ or }
\]

\[
    (s = W) \text{ and } (m \text{ prefix of } m')
\]

Applies to infinite traces.

- \( \leq \) is a partial order.
- \( > \) is a well-founded order.
- \( W[\_] \) is the bottom trace.
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Prefix Closure (downward closure)

Prefix closure of trace $t$ is the set of all its prefixes:

$$t^* = \{ s \mid s \leq t \}$$

For traceset (non-empty set of traces) $p$ define downward closure by:

$$p^* = \bigcup_{t \in p} (t^*)$$, for non-empty $p$

$$(p \times q \times \cdots \times r)^* = p^* \times q^* \cdots \times r^*$$ Cartesian Product
A specification ($\text{spec}$) is a non-empty prefix-closed set of traces, i.e., $p = p^*$. 
Meaning of spec

- Each trace in a spec of $f$ is a possible execution of $f$ in some environment.

- So, a spec is prefix-closed.

- Deadlock: A spec that includes $W[m]$ but no extension.

- Eventual halting:
  - Every waiting trace has an extension by an autonomous event.
  - There is no divergent trace.
Tree depiction of a spec is insufficient

Toss a coin sequentially until it lands Heads.

unfair coin: $\{H[Tails^j, Heads] \mid j \geq 0\} \ast \cup \{D[Tails^\omega]\}$

fair coin: $\{H[Tails^j, Heads] \mid j \geq 0\} \ast$

Explicit inclusion/exclusion of infinite traces in a spec.
Denotational Semantics (repeated)

- $f \oplus g$ is a program constructed out of components $f$ and $g$, and
  
  *combinator* $\oplus$, a programming language construct.

- The specification of $f \oplus g$, $[f \oplus g]$ is given by:
  
  $$[f \oplus g] \Delta [f][\oplus][g]$$

- $[\oplus]$ is a *transformer*:
  
  It combines two specifications, $[f]$ and $[g]$, to yield a specification.

  *Notation Overloading:* use $\oplus$ instead of $[\oplus]$. 
A Motivating Example

- Programming language construct, \( \oplus: \oplus (A, B, C) \)

- Execute \( A, B \) concurrently.

- If \( A \) engages in \( e \) and \( B \) in \( \overline{e} \), they rendezvous. Then start \( C \) to run concurrently with \( A \) and \( B \).
A Motivating Example: $\oplus (A, B, C)$

- Let specifications of $A$, $B$, $C$ be $p$, $q$, $r$, respectively.

- $C'$ starts with event $a$ and then behaves as $C$:
  spec is $\text{cons}(a, r)$.

- spec of $A$, $B$, $C'$ running concurrently: $p \parallel q \parallel \text{cons}(a, r)$.

- Retain those traces in which $\{e, \bar{e}, a\}$ are contiguous.
  Replace these 3 events by event $\tau$:
  $\text{rendezvous}(\{e, \bar{e}, a\}, \tau, (p \parallel q \parallel \text{cons}(a, r)))$

- Drop the $\tau$ symbol from each trace:
  $\oplus' (p, q, r) = \text{drop}(\tau, \text{rendezvous}(\{e, \bar{e}, a\}, \tau, (p \parallel q \parallel \text{cons}(a, r))))$
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- Let specifications of \( A, B, C \) be \( p, q, r \), respectively.

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Example Transformer: Sequential Composition, $f; g$

- $g$ starts executing when and only when $f$ halts.

- A trace of $f; g$ is of the form:
  - $s[m]$ where $s[m]$ is a trace of $f$ and $s$ is $W$ or $D$, or
  - $s[m \; n]$ where
    - $H[m]$ is a trace of $f$
    - $s[n]$ is a trace of $g$
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Example Transformer: parallel composition, $f \mid g$

- $f$ and $g$ execute independently.

- Let $s[m]$ be a trace of $f$, $t[n]$ of $g$, $s$ and $t$ from $\{H, W\}$.

Then, $f \mid g$ includes traces $(s \cap t)(m \otimes n)$ where:

  - $\cap$ symmetric. $H \cap s = s$, $W \cap W = W$.

  - $m \otimes n$ is all interleavings (merge) of $m$ and $n$.

- Merging with infinite sequence: fair and unfair merge.
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- Merging with infinite sequence: \textit{fair} and \textit{unfair} merge.
Definition: Transformer, Trace-wise Transformer

- A transformer is a function that maps a tuple of specs to a spec:
  \[ f(p, q, \cdots, r) \]

  Notation: Infix \( p \oplus q \) for 2-tuple transformer.

- **Tracewise-transformer**: Maps a tuple of traces to a traceset. Then,
  \[
  f(p) = \bigcup \{ f(t) \mid t \in p \}
  \]
  \[
  p \oplus q = \bigcup \{ s \oplus t \mid s \in p, \ t \in q \}
  \]

- Henceforth all transformers are trace-wise.

When is \( f(p) \) a spec given that \( p \) is a spec?
Smooth Transformer

- A smooth transformer preserves prefix closure.

- Smooth Transformer: For any trace $s$,

\[ f^*(s) = f(s^*) \quad \text{(Notation: } f^*(s) \text{ is } (f(s))^*) \]

\[ (s \oplus t)^* = s^* \oplus t^* \]
Properties of smooth transformers

- For smooth $f$ and spec $p$, $f_*(p) = f(p_*)$.

- Follows: A smooth transformer transforms specs to specs.

- Composition of smooth transformers is smooth.

- $f$ is smooth iff
  - $f$ transforms specs to specs, and
  - $f$ is monotonic: $s \leq t \Rightarrow f_*(s) \subseteq f_*(t)$.
Example of Smooth Transformer: choice

- $f$ or $g$: choose to execute either $f$ or $g$

  transformer: $s$ or $t = \{s\} \cup \{t\}$

- $or$ is smooth.
Example of Smooth Transformer: cons

- Append a specific event $a$ as the first event of every trace.

- $\text{cons}(a, W[]) = \{W[], W[a]\}$

  $\text{cons}(a, s[m]) = \{s[a \, m]\}$
• A filter transformer accepts or rejects each trace.

• A *filter* is defined by a predicate $b$ on traces, where
  1. $b(W[])$ holds, and
  2. If $b(t)$ holds then $b(s)$ holds for all prefixes $s$ of $t$.

• A filter transformer accepts all prefixes for which $b$ holds.

$$f(t) = \{s \mid b(s) \land s \leq t\}$$
Examples of Smooth transformers

- unfair merge: \( f \ | \ g \)
- fair merge: \( f \ |'g \)
- rendezvous: merge traces so that events \( e \) and \( e' \) are contiguous.
- sequential composition: \( f \ ; \ g \)
  \[
  H[m] \ ; \ t[n] = \{t[m \ n]\},
  \]
  \[
  s \ ; \ t[n] = \{s\}, \text{ otherwise}
  \]
Fairness

- Coin tosses are fair.

- Fair scheduler: In a multiprocess implementation every process gets to execute eventually.

- A semaphore is granted fairly.

- Any finite interval in time can contain only a finite number of events.
Fairness is a filter transformer

- The transformer accepts all finite traces, accepts the fair infinite traces and rejects the unfair ones.

- Fits the definition of a filter, a smooth transformer.

Example: coin toss forever until Heads appears.

- unfair coin:

  \[ \{H[Tails^j, Heads] \mid j \geq 0\} \cup \{D[Tails^\omega]\} \]

- fair coin: Apply the filter that rejects the infinite sequence of Tails.

  \[ \{H[Tails^j, Heads] \mid j \geq 0\}^* \]
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Shared Resource

• Consider $x.read() \mid x.write(3)$, where local variable $x$ is initialized to 0.

• spec of $x.read()$ includes the trace $H[read(5)]$.

  spec of program $x.write(3)$ is $H[write(3)]^*$

• Applying merge: a trace of $x.read() \mid x.write(3)$ is $H[read(5), write(3)]$, an invalid trace.
Parallel executions may not be independent

- The complete program is

  \[
  \text{int } x = 0 \\
  x.\text{read()} \mid x.\text{write}(3)
  \]

- The declaration “int \( x = 0 \)” induces a filter transformer, \( x.\text{int} \).

  It rejects all traces that are not possible with the resource.

- Given specs \( p \) and \( q \) of \( x.\text{read()} \) and \( x.\text{write}(3) \), spec of

  \[
  \text{int } x = 0 \\
  x.\text{read()} \mid x.\text{write}(3)
  \]

  is \( x.\text{int}(p \mid q) \)
Research Area

• Each shared resource is defined by a filter.

• Each filter is an acceptor of strings, i.e., a formal language.

• So, a shared resource can be specified as a language.

• The language may include infinite strings, say, for strong semaphore.

• I have defined filters for read/write shared variables, write-once variables, channel, weak and strong semaphore
Recursion: Procedure $stut()$

- Toss an unfair coin
  if it lands Heads halt, otherwise call $stut()$.

- Let the spec of $stut()$ be $x$.

- $stut()$ chooses between
  - halting the computation (when toss lands Heads), with spec $H[]$, and
  - event Tails followed by $stut()$, with spec $\text{cons}($Tails, $x$)

- The transformer for choice is set union.

- $x = H[] \cup \text{cons}($Tails, $x$),
  \cup and $\text{cons}$ are smooth.
Recursion: Procedure \textit{stut()} \\

- Toss an unfair coin  
  if it lands Heads halt, otherwise call \textit{stut()}.  

- Let the spec of \textit{stut()} be \textit{x}.  

- \textit{stut()} chooses between  
  - halting the computation (when toss lands Heads), with spec \textit{H[]}, and  
  - event \textit{Tails} followed by \textit{stut()}, with spec \textit{cons(Tails, x)}  

- The transformer for choice is set union.  

- \textit{x} = \textit{H[]} \cup \textit{cons(Tails, x)},  
  \cup \text{ and \textit{cons}} are smooth.
Recursion: Procedure `stut()`

- Toss an unfair coin
  if it lands Heads halt, otherwise call `stut()`.

- Let the spec of `stut()` be `x`.

- `stut()` chooses between
  - halting the computation (when toss lands Heads), with spec `H[]`, and
  - event Tails followed by `stut()`, with spec `cons(Tails, x)`

  - The transformer for choice is set union.

- `x = H[] \cup cons(Tails, x)`,
  `\cup` and `cons` are smooth.
Solutions of recursive equation: \( x = f(x) \)

- Extensively studied in denotational semantics where \( x \), called a point, is from a complete partial order (CPO).
  - There is a partial order \( \subseteq \) in the cpo.
  - There is a bottom element, \( \bot \).
  - Every chain \( x_0 \leq x_1 \ldots \) has a least upper bound (lub) \( y \):
    \[
    x_i \subseteq y \quad \text{upper bound} \\
    y \subseteq z \quad \text{for any upper bound } z.
    \]
- A solution of \( x = f(x) \) is a fixed point of \( f \).

Wanted: the least fixed point, \( \text{lfp}(f) \), according to \( \subseteq \).
Least Fixed-point Theorem

- \( F \) is *continuous* means:
  For every chain \( C \), \( f(\text{lub}(C)) = \text{lub}(f(C)) \).

- **Theorem:** Given \( x = f(x) \) where \( f \) is *continuous*:
  \[
lfp(f) = \text{lub}(f^i(W[\ ]))\]

- That is, with
  \[
x_0 = \bot, \quad x_{i+1} = f(x_i),
  \]
  \[
lfp(f) = \text{lub}(x_0, \ldots, x_i, \ldots)\]
In the current work

Specs form a complete partial order, where

- the order relation is subset order over specs, $lub$ is set union,
- $\bot$ is the $W[]$,
- $f$, a smooth transformer is always continuous.
- Proposition: $\text{lfp}(f)$ is the expected outcome in an execution.
Example: $stut()$

- Recursive equation: $x = H[] \cup cons(Tails, x)$

- $lfp(stut) = \{H[Tails^j] \mid j \geq 0\}^*$

- This is not the correct solution.
  Does not include the infinite trace $D[Tails^\omega]$.

The fixed point should include the limit of all trace chains.
The crux of the problem

• We have ordered arbitrary specs by subset ordering. For a chain of specs $p_0 \subseteq p_1 \ldots$, lub is the union of the $p_i$s.

• Consider only upward-closed specs. For a chain of such specs, the lub is upward-closure of their union.
Upward Closure

• Given trace chain \( C, \ C = t_0 \leq t_1 \ldots \)
  Limit of \( C, \ lim(C) \), the shortest trace that has every \( t_i \) as a prefix.

• Define upward closure of spec \( p \) as
  \( p^* = p \cup \{lim(C) \mid C \text{ a chain in } p\} \)

• Follows: for specs, \((p \times q \cdots \times r)^* = p^* \times q^* \cdots \times r^*\)
least upward-closed fixed point (lufp)

• For recursive equation $x = f(x)$, the least upward-closed fixed point $p$ is a spec such that:
  
  $p = f(p)$ fixed point
  
  $p = p^*$ upward-closed
  
  $p \subseteq q$ for any upward-closed fixed point $q$

Note: $p$ is a spec, so downward-closed.

• $\text{lufp}(f)$ may not exist for arbitrary smooth $f$. 
least upward-closed fixed point (lufp)

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  $p = f(p)$  \hspace{1cm} \text{fixed point}

  $p = p^*$  \hspace{1cm} \text{upward-closed}

  $p \subseteq q$  \hspace{1cm} \text{for any upward-closed fixed point $q$}

Note: $p$ is a spec, so downward-closed.

- $\text{lufp}(f)$ may not exist for arbitrary smooth $f$. 
Bismooth Transformer

- Smooth: \( f(p\star) = f\star(p) \), for any traceset \( p \)

- Bismooth:
  - Smooth (preserve downward-closure)
  - Spec \( p \): \( f(p\star) = f\star(p) \) (preserve upward-closure)

Fairness is smooth but not bismooth.

Unfair merge is bismooth, fair merge only smooth.

Continuous filter is bismooth, discontinuous filter only smooth.

All other transformers seen so far are bismooth.
Bismooth Transformer

- Smooth: \( f(p_\ast) = f_\ast(p) \), for any traceset \( p \)

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Fairness is smooth but not bismooth.

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All other transformers seen so far are bismooth.
Proving Bismoothness

- A transformer $f$ maps a spec $p$, a tree of traces, to $f(p)$, another tree of traces.

- $f$ smooth: maps every finite path $x$ of $p$ to a set of paths in $f(p)$.

- $f$ bismooth: for every path $y$ in $f(p)$ there is path $x$ in $p$ mapping to $y$.

- Use Koenig’s infinity lemma: if $p$ has finite degree, i.e., bounded non-determinism.
A variation of Koenig’s infinity lemma

- $S$ and $T$ rooted trees. $S$ domain spec, $T$ range.

- **cover**: a binary relation over $S \times T$. Corresponds to a transformer from $S$ to $T$.

- Node $x$ of $S$ covers node $y$ of $T$ means $(x, y) \in \text{cover}$. Also, $y$ covered by $x$.

- Nodeset $X$ covers $Y$ ($Y$ covered by $X$): every node of $Y$ covered by some node of $X$. 
Variation, Contd.

Theorem: Given $S$, $T$ and cover as above, suppose:

- Each node of $T$ is covered by a non-empty finite set of nodes of $S$.
- If node $x$ covers node $y$ then the ancestors of $x$ in $S$ (that includes $x$) cover the ancestors of $y$.

Then every path of $T$ is covered by some path of $S$. 
Sufficient Condition for Bismoothness

A transformer is **co-finite** means:

it maps a *finite* number of finite traces to any finite trace.

Theorem: A transformer that is **smooth**, **co-finite** and **chain continuous** is bismooth.
Least Upward-closed Fixed-point of Bismooth Transformer

Theorem: For bismooth $f$, $\text{lufp}(f) = \text{lfp}^*(f)$
Revisit \textit{stut()} \\

- **Recursive equation:** \( x = H[\cdot] \cup \text{cons}(\text{Tails}, x) \) \\

- \( \text{lfp}(\text{stut}) = \{H[\text{Tails}^j] \mid j \geq 0\}^* \) \\

- \( \text{lufp}(\text{stut}()) \) 
  
  = {From theorem} 
  \( \text{lfp}^*(\text{stut}()) \) 
  
  = \{ \text{lfp}(\text{stut}) = \{H[\text{Tails}^j] \mid j \geq 0\}^* \}^* 
  
  (\{H[\text{Tails}^j] \mid j \geq 0\}^*)^* 
  
  = {computing} 
  
  \{H[\text{Tails}^j] \mid j \geq 0\}^* \cup \{D[\text{Tails}^\omega]\} \)
Fairness and Recursion

- Let \( x = f(x) \) where \( f \) is smooth, not bismooth.

- \( f \) may have no upward-closed fixed point.

- maximal fixed-point: one that includes as many limit traces as possible (under the fairness constraint).

- the min-max fixed-point, \( mmfp(f) \): the least maximal fixed-point.

**Theorem:** \( mmfp(f) = \) the greatest fixed point of \( f \) in \( lfp^*(f) \).