Arithmetic Mean is greater than or equal to Geometric Mean
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Let B be a finite non-empty bag of positive real numbers. Let $a(B)$ and $g(B)$ denote its arithmetic and geometric means, respectively.

Theorem 1: $a(B) \geq g(B)$.

The following proof is based on a proof from *Inequalities* by Beckenbach and Bellman, section 11.

Let r be any positive real number and B' be the bag obtained by multiplying each number in B by r. Then, from the definitions of arithmetic and geometric means, $a(B') = r \cdot a(B)$ and $g(B') = r \cdot g(B)$. That is, $a(B) \geq g(B) \equiv a(B') \geq g(B')$. Therefore, it suffices to prove the theorem after scaling the numbers in B by any such r. We assume that the numbers in B have been so scaled that the product of all numbers in B is 1; hence, $g(B) = 1$. We show that $a(B) \geq 1$, or that the sum of the numbers in B is at least the size of B. This is the statement of Theorem 2.

Theorem 2: Let B be a finite non-empty bag of n positive real numbers whose product is 1. Then, $sum(B) \geq n$, where $sum(B)$ is the sum of the numbers in B.

Proof: Proof is by induction on n, the size of B. For $n = 1$, the result is obvious. Consider a bag, B, that has $n+1$ elements, $n \geq 1$. Let x be a smallest and y a largest element, x and y being distinct elements of the bag. Since the product of the elements is 1, $x \leq 1$, and, similarly, $y \geq 1$. Let C be the bag obtained by replacing x and y in B by their product, i.e., $C = B - \{x\} - \{y\} + \{x \times y\}$. Product of the numbers in C is 1, and according to the induction hypothesis, $sum(C) \geq n$.

$$
\begin{array}{cl}
 & sum(B) \\
= & \{\text{Definition of } C\} \\
 & sum(C) + x + y - x \times y \\
\geq & \{x \leq 1,\ y \geq 1 \Rightarrow (1-x)(y-1) \geq 0, \text{ or } x + y - x \times y \geq 1\} \\
 & sum(C) + 1 \\
\geq & \{sum(C) \geq n, \text{ from induction hypothesis}\} \\
 & n + 1
\end{array}
$$

Note Added; 9/28/06 The following proof of Theorem 1, which appears in Hardy, Littlewood and Polya, was shown to me by Anindya Patthak. First, prove the result for all bags whose sizes are powers of 2. Then show that if the result holds for all bags of size n, $n > 1$, it holds for all bags of size $n-1$ as well.

• Proof for bags whose sizes are powers of 2: The result holds trivially for bags of size 2^0.

For a bag of size 2, say with elements x and y:

$$
\begin{array}{ll}
 & (x+y)^2 \\
= & \{\text{algebra}\} \\
 & (x-y)^2 + 4xy \\
\geq & \{(x-y)^2 \geq 0\} \\
 & 4xy
\end{array}
$$

It follows that $1/2(x+y) \geq \sqrt{xy}$.

For a bag of size 2^n, $n > 1$:

Divide the bag into two bags of equal sizes whose arithemetic means are a and a' and geometric means are g and g', respectively. Inductively, $a \geq g$ and $a' \geq g'$. The arithmetic mean of the original bag is $1/2(a+a')$ and the geometric mean is $\sqrt{gg'}$.

$$
\begin{array}{ll}
 & 1/2(a+a') \\
\geq & \{a \geq g \text{ and } a' \geq g'\} \\
 & 1/2(g+g') \\
\geq & \{\text{consider the bag } \{g, g'\}\text{; from the last proof}\} \\
 & \sqrt{gg'}
\end{array}
$$

• Given that the result holds for all bags of size n, it holds for all bags of size $n-1$:

Given is a bag B of size $n-1$. Let the sum of its elements be A, product be G, arithmetic mean be a and geometric mean be g. Therefore, $a = (1/(n-1))A$ and $g = G^{1/(n-1)}$. Consider the bag $B' = B \cup \{a\}$. The sum and product of B' are $A+a$ and Ga, respectively.

First, observe that the arithmetic mean of B' is a, because $(1/n)(A+a) = (1/n)(A + (1/(n-1))A) = (1/n)(n/(n-1))A = (1/(n-1))A = a$. And, the geometric mean of B' is $(Ga)^{1/n}$.

From the inductive hypothesis, for B'

$$
\begin{array}{ll}
 & a \geq (Ga)^{1/n} \\
\Rightarrow & \{\text{algebra}\} \\
 & a^n \geq Ga \\
\Rightarrow & \{\text{algebra}\} \\
 & a^{n-1} \geq G \\
\Rightarrow & \{\text{algebra}\} \\
 & a \geq G^{1/(n-1)} \\
\Rightarrow & \{g = G^{1/(n-1)}\} \\
 & a \geq g
\end{array}
$$