A Problem due to J Moore
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5/22/2025, supersedes a note of 3/28/01

1 The Problem

The following problem was posed by J Moore during the faculty lunch on March 28, 2001. This note supersedes a note I wrote on that day.

There are two machines –α and β– with two registers each. Each machine can read/write a shared counter. Initially the counter holds the value 1 and all registers are empty, i.e. have a value of 0. There are two atomic actions (steps):

1. (read) Any machine may read the counter value into one of its empty registers (and then the register becomes nonempty).
2. (write) A machine may write the sum of its register values into the counter and then empty both registers.

Moore asked for the set of counter values that can be generated. I show two different schedules to generate all possible integers.

2 Schedules

The state of the system is a triple $\langle a, b, c\rangle$, where

a is the set of register values of α
b is the set of register values of β
c is the value in the counter.

The initial state is $\langle\{\}, \{\}, 1\rangle$. In a proof, $\langle a, b, c\rangle \rightarrow \{step\}\langle a', b', c'\rangle$ denotes that the atomic action (explained in *step*) causes the transition from state $\langle a, b, c\rangle$ to $\langle a', b', c'\rangle$. The symbols $\stackrel{*}{\rightarrow}$ and $\stackrel{**}{\rightarrow}$, in place of $\rightarrow$, denote, respectively, a bounded number of steps, i.e. in $\mathcal{O}(1)$ steps, and a finite number of steps, not necessarily bounded.

Lemma 1

1. $\langle\{\}, \{1\}, k\rangle\rangle \quad \stackrel{*}{\rightarrow} \quad \langle\{\}, \{1\}, k+1\rangle$.
2. $\langle\{\}, \{1\}, k\rangle \quad \stackrel{*}{\rightarrow} \quad \langle\{\}, \{1\}, 2k\rangle$.

Proof:

1.
$$\begin{array}{ll} & \langle\{\}, \{1\}, k\rangle \\ \rightarrow & \{\alpha \text{ reads}\} \\ & \langle\{k\}, \{1\}, k\rangle \\ \rightarrow & \{\beta \text{ writes}\} \end{array}$$

$$
\begin{aligned}
& \langle \{k\}, \{\}, 1\rangle \\
\rightarrow\ & \{\alpha \text{ reads}\} \\
& \langle \{k, 1\}, \{\}, 1\rangle \\
\rightarrow\ & \{\beta \text{ reads}\} \\
& \langle \{k, 1\}, \{1\}, 1\rangle \\
\rightarrow\ & \{\alpha \text{ writes}\} \\
& \langle \{\}, \{1\}, k+1\rangle
\end{aligned}
$$

2.

$$
\begin{aligned}
& \langle \{\}, \{1\}, k\rangle \\
\rightarrow\ & \{\alpha \text{ reads}\} \\
& \langle \{k\}, \{1\}, k\rangle \\
\rightarrow\ & \{\alpha \text{ reads}\} \\
& \langle \{k, k\}, \{1\}, k\rangle \\
\rightarrow\ & \{\alpha \text{ writes}\} \\
& \langle \{\}, \{1\}, 2k\rangle
\end{aligned}
$$

□

Next, I show that (1) all positive integers up to n can be enumerated in $\mathcal{O}(n)$ steps, and (2) a specific n in $\mathcal{O}(\log n)$ steps.

Theorem 1

1. $\langle \{\}, \{\}, 1\rangle \overset{**}{\rightarrow} \langle \{\}, \{\}, k$, for all k, $k \leq n$, in $\mathcal{O}(n)$ steps.
2. $\langle \{\}, \{\}, 1\rangle \overset{**}{\rightarrow} \langle \{\}, \{\}, n$, in $\mathcal{O}(\log n)$ steps.

Proof: The proof is by induction on n.

Base Case, $n = 1$**:** I show that $\langle \{\}, \{1\}, 1\rangle$ is reachable in $\mathcal{O}(1)$ steps.

$$
\begin{aligned}
& \langle \{\}, \{\}, 1\rangle \\
\rightarrow\ & \{\beta \text{ reads}\} \\
& \langle \{\}, \{1\}, 1\rangle
\end{aligned}
$$

1. Follows directly from Lemma 1 part(1).
2. I show the result for even and odd values of n separately.

   Case $n = 2k$**, for some** k**,** $k > 0$**:** Inductively, $\langle \{\}, \{1\}, k\rangle$ is reachable in $\mathcal{O}(\log k)$ steps. From Lemma 1 part(2),

   $\langle \{\}, \{1\}, k\rangle \overset{*}{\rightarrow} \langle \{\}, \{1\}, 2k\rangle$. Therefore,

   $\langle \{\}, \{1\}, 2k\rangle$ is reachable in $\mathcal{O}(\log k) = \mathcal{O}(\log n)$ steps.

   Case $n = 2k+1$**, for some** k**,** $k > 0$**:** Inductively, $\langle \{\}, \{1\}, 2k\rangle$ is reachable in $\mathcal{O}(\log k)$ steps. From Lemma 1 part(1),

   $\langle \{\}, \{1\}, 2k\rangle \overset{*}{\rightarrow} \langle \{\}, \{1\}, 2k+1\rangle$. Therefore,

   $\langle \{\}, \{1\}, 2k+1\rangle$ is reachable in $\mathcal{O}(\log k) = \mathcal{O}(\log n)$ steps. □