Chameleons
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Problem Description The following problem, involving multicolored chameleons, appears in the May 2009 issue of CACM. Given below is a colorless description that captures the essentials.

A game starts with three piles of chips. A step of the game consists of removing one chip each from two different piles and adding both chips to the third pile. The game terminates (reaches a final state) when no more steps can be taken, i.e., there are at least two empty piles. Devise a strategy to arrive at a final state, or prove that no such strategy exists for the given initial state.

Necessary Condition for a Solution Denote the number of chips in the three piles by p, q, and r at any point during game. Also call the piles themselves p, q and r when there can be no confusion. A move, say, from p and q to r decreases both p and q by 1 each, so $p-q$ is unchanged. And, $r-p$ (and $r-q$) is increased by 3 because r increases by 2 and p (and q) decreases by 1. In either case, the difference modulo 3 between any two piles is unchanged by a move.

The game terminates only when there are at least two empty piles, say, $p = 0$ and $q = 0$; so, $p \overset{\text{mod } 3}{\equiv} q$. Since a move does not affect $p \overset{\text{mod } 3}{\equiv} q$, it holds initially. Thus, a necessary condition for the termination of the game is that initially some pair of piles are congruent modulo 3.

Sufficient Condition for a Solution We show that the given necessary condition is also sufficient. Initially, let p and q be piles such that $p \leq q$ and $p \overset{\text{mod } 3}{\equiv} q$. We prove that any state in which $p \leq q$ and $p \overset{\text{mod } 3}{\equiv} q$ holds is either a final state or there is a move that decreases q and retains $p \leq q$ and $p \overset{\text{mod } 3}{\equiv} q$. Since q is always non-negative, the number of moves is bounded by the initial value of q; so, the game terminates and the resulting state then is a final state.

If $q = 0$ in any state, from $p \geq 0$ and $p \leq q$, $p = 0$. So, both p and q are empty, and this is a final state. Now suppose that $q > 0$.

Case 1) $p \neq 0$: Move a chip from each of p and q to r. This preserves $p \leq q$ and $p \overset{\text{mod } 3}{\equiv} q$, and decreases q.

Case 2) $p = 0$: If $r = 0$, this is a final state. So, assume that $r > 0$. From $q > 0$ and $p \overset{\text{mod } 3}{\equiv} q$, $q \geq 3$. Move a chip from each of q and r to p. In the resulting state $p = 2$ and $q \geq 2$. So, $p \leq q$ and, as we have argued earlier, $p \overset{\text{mod } 3}{\equiv} q$. The move decreases q.

A Small Generalization Suppose there are n piles, $n \geq 3$, where a move removes one chip from all but one pile and puts those $n-1$ chips in the remaining pile. The termination condition is identical; if there are two or more empty piles a move becomes impossible and the game terminates. As before, the difference modulo n between any two piles remains unchanged by a move. So the necessary condition is similar to that in the last case with n replacing 3. This is also a sufficient condition, using a nearly identical proof.

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