Common Meeting Time

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Let $f(t)$ be the earliiest time at/after t when f can meet. From this description,

1. $f(t)$ is at or after t: $f(t) \geq t$.
2. f can meet at $f(t)$: $f(f(t)) = f(t)$.
3. $f(t)$ is the earliest time at or after t when f can meet: $t \leq s \Rightarrow f(t) \leq f(s)$.

To see (3), note that from $t \leq s$: (4) $t \leq s \leq$ {from 1} $f(s)$, and (5) $f(f(s)) = f(s)$. From (5), $f(s)$ is a time at which f can meet and it is at/after t from (4). Since $f(t)$ is the earliest meeting time at/after t, $f(t) \leq f(s)$.

Observe that the conditions given above are exactly those of a closure, C, of a set, S: (1) $S \subseteq C(S)$, (2) $C(C(S)) = C(S)$, and (3) $S \subseteq T \Rightarrow C(S) \subseteq C(T)$.

For the solution we don't need (2). If we use (1) only, we may discover a meeting time, not necessarily the earliest. If we use (2) only, the times could decrease and there is no guarantee of convergence.

Observe that (1) does not imply (3): $f(1) = 5, f(2) = 3$. Nor does (3) imply (1): For all $t, f(t) = 1$.

A sequential Algorithm: Let there be N functions, f_0 through f_{N-1}. Let v_i denote a value held by the i^{th} process.

begin
  $t := 0$; $i := 0$; $(\forall i :: v_i = 0)$;
  while $t \neq v_i$ do
    if $f_i(t) \neq t$ then $\{f_i(t) > t\} t, v_i := f_i(t), f_i(t)$ fi;
    $i := i + 1$ {this is addition mod N}
  od {Report t as the earliest common meeting time}
end

To show that on termination of the loop, $(\forall j :: f_j(t) = t)$, use the invariant, for all j

$$t \geq v_j \wedge [v_j = t \Rightarrow (\forall k : k \in j..i : f_k(t) = t)] \wedge [\forall s : s < t : \exists r : f_r(s) \neq s].$$

Here, $j..i$ is the set consisting of the indices $j, j+1, .., i-1$, where all arithmetic is mod N; note that $j..j$ includes all the process indices, and $j..(j+1)$ includes only j.