Russell Paradox, Cantor Diagonalization
Jayadev Misra
9/28/99

Russell Paradox Let S be a set defined as follows:

$S = \{x \mid x \notin x\}$.

It is required to show that S is not a well-defined set. From its definition, for all z,

$z \in S \equiv z \notin z$.

Setting z to S, we have

$S \in S \equiv S \notin S$,

thus deriving a contradiction.

Cantor's Diagonalization It is required to show that for any set its powerset is strictly larger. The idea is to show that there is no 1-1 function from 2^S to S, for any S. Our arguments apply for any set, finite or infinite.

- (Indirect Proof) Since S is no larger than 2^S, it is sufficient to show that there is no 1-1 correspondence between S and 2^S, i.e., for any two functions f, g, $f : S \longrightarrow 2^S$ and $g : 2^S \longrightarrow S$, f, g are not inverses of each other.

  Define

  $X = \{z \mid z \in S \ \wedge \ z \notin f(z)\}$, i.e., for any z in S
  $(z \in X) \equiv (z \notin f(z))$.

  Since X is a subset of S, $X \in 2^S$. Instantiating z by $g(X)$ above,

  $$\begin{array}{ll} & [g(X) \in X] \equiv [g(X) \notin f(g(X))] \\ \Rightarrow & \{\text{Predicate Calculus}\} \\ & f(g(X)) \neq X \\ = & \{\text{Definition of function inverse}\} \\ & f, g \text{ are not inverses} \end{array}$$
  $\Box$

- (Direct Proof) Let g be a function from 2^S to S. We show that g is not 1-1. Define X by $(\forall y :: [y \in X] \equiv [\exists Z : g(Z) = y : y \notin Z])$. Since X is a subset of S, $X \in 2^S$.

  $$\begin{array}{ll} & [g(X) \in X] \not\equiv [g(X) \notin X] \\ = & \{\text{Instantiating } y \text{ by } g(X) \text{ in the definition of } X\} \\ & [\exists Z : g(Z) = g(X) : g(X) \notin Z] \not\equiv [g(X) \notin X] \\ = & \{\text{one-point rule on the second clause}\} \\ & [\exists Z : g(Z) = g(X) : g(X) \notin Z] \not\equiv [\exists Z : Z = X : g(X) \notin Z] \\ \Rightarrow & \{\text{Predicate Calculus}\} \\ & (\exists Z : [Z = X] \not\equiv [g(Z) = g(X)]). \end{array}$$

Hence g is not a 1-1 function. $\square$

Note: The definition of X in the Direct Proof could be changed to $(\forall y :: [y \in X] \equiv [\forall Z : g(Z) = y : y \notin Z])$. The proof still goes through because the one-point rule works just as well with universal quantification. I owe this observation to E.W. Dijkstra, along with a much cleaner version of the Direct Proof.