Dijkstra's Proof of Hall's Theorem

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Let F a finite family of subsets of elements (*family* means multiset in this context; so, members of F may be identical). F has a *system of distinct representatives* (abbreviated by *SDR*) if it is possible to choose an element from each member of F so that all chosen elements are distinct.

Hall's Theorem[3]: An *SDR* exists for F if and only if union of any k members of F has at least k elements, for all k, $1 \leq k \leq |F|$.

The necessity of the condition is obvious, but sufficiency is much harder to prove; see [1] for a proof. A proof using the theory of maximum flows appears in [2], Chapter II, section 10. The following Proof is due to Dijkstra.

Call each element a *color*, a set of colors is a *group*. A set of groups *cover* the colors in those groups. A set of k groups is *happy* if the groups cover at least k distinct colors.

Proof of Hall's Theorem: The proof is by induction on N, the number of groups in F. For $N = 1$, from the Hall condition, there a single group that covers at least one color which may serve as its representative.

For $N > 1$: let x be a color covered by group r; suppose removal of x from r does not cause F to violate Hall-Condition. Then, we may remove x from r and find a set of representatives for the (reduced) groups. Eventually, we will find x and r such that removal of x from r causes F to violate Hall-Condition.

Lemma 1: There is a subset G of k groups of F, $1 \leq k < N$, that covers exactly k colors.
Proof: Let r' be the group obtained from r by removing x from it. Let R be a subset of groups such that $R \cup r'$ is unhappy. If R is empty then let G consist of the single group r, and $k = 1$. Otherwise, let G be R, where R has k groups. We show that G covers exactly k colors: (1) R covers at least k colors, from the Hall condition, (2) hence, $R \cup r'$ covers at least k colors, (3) $R \cup r'$ covers less than $k + 1$ colors, from the assumption that $R \cup r'$ is unhappy. Therefore, R (and $R \cup r'$) covers k colors. Observe that in both cases (R empty and R non-empty), $1 \leq k < N$.

Observation 1: Every subset in G is happy.
Proof: $G \subseteq F$ and every subset in F is happy.

Definition: Let the colors covered by G be called *primary*, and the remaining colors be *secondary*. Let H be $F-G$. Note that H is non-empty since $1 \leq k < N$ (and $N > 1$).

Lemma 2: Any m groups from H cover at least m secondary colors.
Proof: Consider a subset S of m groups from H. $S \cup G$ covers at least $m + k$ colors, from the Hall condition. Since there are k primary colors, $S \cup G$ covers at least m secondary colors. Since G covers no secondary color, S covers at least m secondary colors.

Now, apply the construction recursively on G and H both of which are non-empty, restricting the selection of colors to secondary colors in H.

References

[1] Béla Bollobás. *Graph Theory.* Springer-Verlag, 1979.

[2] L.R Ford, Jr. and D.R. Fulkerson. *Flows in Networks.* Princeton University Press, 1962.

[3] P. Hall. On Representatives of Subsets. *Journal of London Mathematical Society*, 10:26–30, 1935.