Call an element a \textit{color} and a set of colors a \textit{group}. Let $F$ be a set of groups. Suppose $F$ meets the Condition in Hall’s theorem for Distinct Representatives, i.e., every subset of $F$ consisting of $k$ groups has at least $k$ colors. Let $g$ be a group such that $F \cup \{g\}$ does not meet the Hall condition. We give a characterization of the subsets of $F$ which together with $g$ fail to satisfy Hall-Condition.

\textbf{Definition:} A subset $G$ of $F$ is a \textit{hit} if $G \cup \{g\}$ has fewer colors than $|G| + 1$.

A hit $G$ is a subset of $F$; hence, it meets the Hall-condition. Therefore, the number of colors in $G$ is $\geq |G|$. From the definition of hit, the number of colors in $G$ equals $|G|$.

\textbf{Theorem:} The hits form a complete lattice.

\textbf{Proof:} Let $P, Q, R$ be three disjoint subsets of $F$ where $P \cup Q$ and $Q \cup R$ are hits. We show that $Q$ is a hit and so is $P \cup Q \cup R$. Let $P, Q, R$ have $p$, $q$, $r$ groups in them respectively. We write $PQ$ for $P \cup Q$, $QR$ for $Q \cup R$ and $PQR$ for $P \cup Q \cup R$. Let $cP$ denote the set of colors in $P$. The fact that $F$ meets Hall-Condition implies that every subset of $F$ has at least as many colors as groups, i.e., for $G$, a subset of $F$, $|G| \leq |cG|$. Since $PQ$ and $QR$ are hits $|cPQ| = p + q$, $|cQR| = q + r$.

$$\begin{align*}
\text{Number of groups in } PQR &= \{|\text{disjointness of } P, Q, R\} \\
&= p + q + r \\
&\leq \{|\text{PQR meets Hall-Condition}\} \\
&= \{|cPQR| = |cPQ \cup cQR|, \text{ set theory}\} \\
&= |cPQ \cup cQR|
\end{align*}$$
\[ |A \cup B| = |A| + |B| - |A \cap B|, \text{ set theory} \]
\[ |cPQ| + |cQR| - |cPQ \cap cQR| \]
\[ = \{cPQ \cap cQR = cQ \cup (cP \cap cR) - cQ\}, \text{ set theory} \]
\[ |cPQ| + |cQR| - |cQ| - |(cP \cap cR) - cQ| \]
\[ = \{ |cPQ| = p + q, \ |cQR| = q + r; \text{ using } b \text{ for } |(cP \cap cR) - cQ| \} \]
\[ p + q + q + r - |cQ| - b \]

We conclude that
\[ p + q + r \leq p + q + q + r - |cQ| - b, \text{ or} \]
\[ 0 \leq q - |cQ| - b, \text{ or} \]
\[ |cQ| + b \leq q \]

Now, since \( Q \) meets Hall-Condition,
\[ q \leq |cQ|. \text{ Therefore,} \]
\[ |cQ| + b \leq q \leq |cQ| \]
Since all quantities are natural, \( |cQ| = q \) and \( b = 0 \). Hence, number of colors of \( PQR \) is \( p + q + r \). Also,

\[ cg \subseteq cPQ, \text{ and} \]
\[ cg \subseteq cQR. \text{ Hence,} \]
\[ cg \subseteq cPQ \cap cQR. \text{ That is,} \]
\[ cg \subseteq cQ \cup (cP \cap cR) - cQ \}. \text{ Since } |(cP \cap cR) - cQ| = b = 0 \]
\[ cg \subseteq cQ \]

Since the number of groups in \( Q \) equals the number of colors in \( Q \) and colors of \( g \subseteq \text{colors of } Q \), \( Q \) is a hit.

Next, observe that \( cg \subseteq cPQ \subseteq cPQR \), and number of groups in \( PQR \) equals the number of colors in \( PQR \). Hence \( PQR \) is also a hit.