Remarks on Hall's Theorem of Distinct Representatives

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Call an element a *color* and a set of colors a *group*. Let F be a set of groups. Suppose F meets the Condition in Hall's theorem for Distinct Representatives, i.e., every subset of F consisting of k groups has at least k colors. Let g be a group such that $F \cup \{g\}$ does not meet the Hall condition. We give a characterization of the subsets of F which together with g fail to satisfy Hall-Condition.

Definition: A subset G of F is a *hit* if $G \cup \{g\}$ has fewer colors than $|G| + 1$.

A hit G is a subset of F; hence, it meets the Hall-condition. Therefore, the number of colors in G is $\geq |G|$. From the definition of hit, the number of colors in G equals $|G|$.

Theorem: The hits form a complete lattice.

Proof: Let P, Q, R be three disjoint subsets of F where $P \cup Q$ and $Q \cup R$ are hits. We show that Q is a hit and so is $P \cup Q \cup R$. Let P, Q, R have p, q, r groups in them respectively. We write PQ for $P \cup Q$, QR for $Q \cup R$ and PQR for $P \cup Q \cup R$. Let cP denote the set of colors in P. The fact that F meets Hall-Condition implies that every subset of F has at least as many colors as groups, i.e., for G, a subset of F, $|G| \leq |cG|$. Since PQ and QR are hits $|cPQ| = p + q$, $|cQR| = q + r$.

Number of groups in PQR
$= \{$disjointness of P, Q, $R\}$
$p + q + r$
$\leq \{PQR$ meets Hall-Condition$\}$
$|cPQR|$
$= \{cPQR = cPQ \cup cQR$, set theory$\}$
$|cPQ \cup cQR|$

$= \{|A \cup B| = |A| + |B| - |A \cap B|$, set theory$\}$
$|cPQ| + |cQR| - |cPQ \cap cQR|$
$= \{cPQ \cap cQR = cQ \cup ((cP \cap cR) - cQ)$, set theory$\}$
$|cPQ| + |cQR| - |cQ| - |(cP \cap cR) - cQ|$
$= \{|cPQ| = p + q,\ |cQR| = q + r$; using b for $|(cP \cap cR) - cQ|\}$
$p + q + q + r - |cQ| - b$

We conclude that
$p + q + r \leq p + q + q + r - |cQ| - b$, or
$0 \leq q - |cQ| - b$, or
$|cQ| + b \leq q$
Now, since Q meets Hall-Condition,
$q \leq |cQ|$. Therefore,
$|cQ| + b \leq q \leq |cQ|$.
Since all quantities are natural, $|cQ| = q$ and $b = 0$. Hence, number of colors of PQR is $p + q + r$. Also,

$cg \subseteq cPQ$, and
$cg \subseteq cQR$. Hence,
$cg \subseteq cPQ \cap cQR$. That is,
$cg \subseteq cQ\ \cup ((cP \cap cR) - cQ)$. Since $|(cP \cap cR) - cQ| = b = 0$
$cg \subseteq cQ$.
Since the number of groups in Q equals the number of colors in Q and colors of $g \subseteq$ colors of Q, Q is a hit.

Next, observe that $cg \subseteq cPQ \subseteq cPQR$, and number of groups in PQR equals the number of colors in PQR. Hence PQR is also a hit.