A Proof about the Harmonic Series
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The following problem and its solution was shown to me by Carroll Morgan.

Problem: Show that $\sum_{i=2}^{n} \frac{1}{i}$ is non-integer, for all n, $n \geq 2$.

Proof: Let p be the largest prime less than or equal to n. Such a p exists because $n \geq 2$.

Lemma 1: $n! = k \times p$ for some k where $k \neq 0 \bmod p$.

Proof: from Bertrand's theorem, $2 \times p > n$. Therefore, $n!$ has exactly one factor –p– that is a multiple of p.

Lemma 2: For all i, $2 \leq i \leq n$: $(\frac{n!}{i} = 0 \bmod p) \equiv (i \neq p)$.

Proof: Using Lemma 1,
For $i = p$, $(\frac{n!}{p} = k \neq 0 \bmod p)$.
For $i \neq p$, $(\frac{n!}{i} = \frac{k}{i} \times p = 0 \bmod p)$, since k is divisible by i.

The given sum is,

$$\frac{\sum_{i=2}^{n} \frac{n!}{i}}{n!}$$

The numerator is a sum of terms all of which except one is congruent to $0 \bmod p$. Therefore, numerator $\neq 0 \bmod p$. Hence, the numerator is not divisible by $n!$ since the latter contains p as a factor.