A Property of the Identity Function: An Exercise in Induction

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Let f be a function from naturals to naturals. It is given that

Property **P**:: $(\forall n :: f^2(n) < f(n+1))$.

Prove that f is the identity function. We will actually prove that the same result holds under the more general condition given below:

Property **Q**:: $(\forall n : (\exists i : i \geq 2 : f^i(n) < f(n+1)))$.

I have heard that this problem appeared in a mathematical olympiad. The problem was shown to me by van de Snepscheut on 12/13/89, who received it from Richard Bird. This is a belated recording of my response to van de Snepscheut, though the generalization is new.

Henceforth, all variables are naturals.

Lemma 1: f is increasing, i.e., $(\forall n :: n \leq f(n))$.
Proof:: There seems to be no direct proof of this result by induction. We will show, instead, that $R ::\ (\forall n :: R.n)$, where $R.n$ is $(\forall t :: n \leq f(n+t))$. The desired result follows by setting t to 0 in each $R.n$. The proof of R is by induction on n.

$R.0 ::\ (\forall t :: 0 \leq f(t))$. This follows because f is a function from naturals to naturals.
$R.n \Rightarrow R.(n+1)$: We prove $n+1 \leq f(n+1+t)$ for arbitrary t, assuming that $R.n$ holds.

$\quad$ *true*
$\Rightarrow$ {Induction hypothesis}
$\quad$ $(\forall s :: n \leq f(n+s)) \wedge n \leq f(n+t)$
$\Rightarrow$ {$f(n+t) - n \geq 0$ from above.
$\quad$ Set s to $f(n+t) - n$ in the first term.}
$\quad$ $n \leq f(n + f(n+t) - n)$
$\Rightarrow$ {Rewriting}
$\quad$ $n \leq f^2(n+t)$
$\Rightarrow$ {From property P: $f^2(n+t) \leq f(n+1+t)$}
$\quad$ $n < f(n+1+t)$
$\Rightarrow$ {arithmetic}
$\quad$ $n+1 \leq f(n+1+t)$ $\square$

Lemma 2: f is monotone, i.e., $(\forall m, n :: m \leq n \Rightarrow f(m) \leq f(n))$.
Proof::

$$\begin{array}{ll} & true \\ \Rightarrow & \{\text{Set } n \text{ to } f(n) \text{ in Lemma 1}\} \\ & f(n) \leq f^2(n) \\ \Rightarrow & \{\text{From property P: } f^2(n) < f(n+1)\} \\ & f(n) < f(n+1) \\ \Rightarrow & \{\text{Induction on naturals}\} \\ & m \leq n \Rightarrow f(m) \leq f(n) \end{array}$$

□

Corollary:: $f(n) < f(m) \Rightarrow n < m$, by taking contrapositive of Lemma 2.

Theorem 1: f is the identity function, i.e., $f(n) = n$, for all n.
Proof::

$$\begin{array}{ll} & true \\ \Rightarrow & \{\text{Property P}\} \\ & f(f(n)) < f(n+1) \\ \Rightarrow & \{\text{Corollary of Lemma 2}\} \\ & f(n) < n+1 \\ \Rightarrow & \{\text{Lemma 1: } n \leq f(n)\} \\ & n \leq f(n) < n+1 \\ \Rightarrow & \{\text{Arithmetic}\} \\ & n = f(n) \end{array}$$

A Generalization

We show that if property **Q**:: $(\forall n : (\exists i : i \geq 2 : f^i(n) < f(n+1)))$ holds then f is an identity function. Note that if $i = 0$ for all n then the property is a tautology, $n < n + 1$. For $i = 1$ the conclusion is incorrect; the successor function satisfies the property.

Lemma 3: f is increasing, i.e., $(\forall n :: n \leq f(n))$.
Proof:: Let $S = (\forall n :: S.n)$ where $S.n = (\forall t :: n \leq f(n+t))$. We prove S is by induction on n.

$S.0$:: $(\forall t :: 0 \leq f(t))$. Follows trivially.

$S.n \Rightarrow S.(n+1)$:: By induction hypothesis assume that A:: $(\forall s :: n \leq f(n+s))$.

Claim For all natural k, t, we have $n \leq f^k(n+t)$. Proof is by induction on k.

$k = 0$: $n \leq n + t$. Follows trivially.

$k + 1$:

$$\begin{array}{ll} & true \\ \Rightarrow & \{\text{Assumption A}\} \\ & n \leq f(n+s) \\ \Rightarrow & \{\text{Induction hypothesis: } n \leq f^k(n+t). \\ & \quad \text{Set } s \text{ to } f^k(n+t) - n\text{; note } s \geq 0.\} \end{array}$$

$$
\begin{array}{ll}
 & n \le f(n + f^k(n+t) - n) \\
\Rightarrow & \{\text{arithmetic}\} \\
 & n \le f^{k+1}(n+t)
\end{array}
\qquad \square
$$

Now we show that $n+1 \le f(n+1+t)$, for any t. For the given n, t, let j be such that $f^j(n+t) < f(n+1+t)$; such a j exists from Property Q.

$$
\begin{array}{ll}
 & \textit{true} \\
\Rightarrow & \{\text{Claim above}\} \\
 & n \le f^j(n+t) \\
\Rightarrow & \{\text{given that } f^j(n+t) < f(n+1+t)\} \\
 & n < f(n+1+t) \\
\Rightarrow & \{\text{arithmetic}\} \\
 & n+1 \le f(n+1+t)
\end{array}
\qquad \square
$$

Corollary: For any natural k, n, we have $n \le f^k(n)$. Proof is by induction on k.

Lemma 4: f is monotone; i.e., $m \le n \Rightarrow f(m) \le f(n)$.
Proof:: Let m be an arbitrary natural. Let i be such that $f^i(m) < f(m+1)$; such an i exists from Property Q.

$$
\begin{array}{ll}
 & \textit{true} \\
\Rightarrow & \{\text{Let } n, k := f(m), i-1 \text{ in Corollary to Lemma 3.} \\
 & \quad \text{Note } i \ge 2 \Rightarrow k \ge 0.\} \\
 & f(m) \le f^{i-1}(f(m)) \\
\Rightarrow & \{\text{Given } f^i(m) < f(m+1)\} \\
 & f(m) < f(m+1)
\end{array}
$$

The result follows by induction on natural numbers. $\square$

Corollary 1:: $f(n) < f(m) \Rightarrow n < m$.
Corollary 2:: For any k, $k \ge 0$, and all m, n, we have $f^k(n) < f^k(m) \Rightarrow n < m$. Proof is by induction on k.

Theorem 2: $f(n) = n$, for all n. Pick an arbitrary n and let $f^i(n) < f(n+1)$.

$$
\begin{array}{ll}
 & \textit{true} \\
\Rightarrow & \{\text{assumption}\} \\
 & f^i(n) < f(n+1) \\
\Rightarrow & \{\text{In corollary to Lemma 3 let } n, k := f(n+1), i-2. \\
 & \quad \text{Note } i \ge 2 \Rightarrow k \ge 0.\} \\
 & f^i(n) < f(n+1) \wedge f(n+1) \le f^{i-2}(f(n+1)) \\
\Rightarrow & \{\text{arithmetic}\} \\
 & f^i(n) < f^{i-2}(f(n+1)) \\
\Rightarrow & \{\text{Rewrite above}\} \\
 & f^{i-1}(f(n)) < f^{i-1}(n+1) \\
\Rightarrow & \{\text{Corollary 2 of Lemma 4 with } k, n, m := i-1, f(n), n+1\}
\end{array}
$$

$$
\begin{array}{ll}
 & f(n) < n+1 \\
\Rightarrow & \{\text{Lemma 3}\} \\
 & n \leq f(n) < n+1 \\
\Rightarrow & \{\text{arithmetic}\} \\
 & f(n) = n
\end{array}
$$

□