**Mapping among Infinite Trees, A Variation of Koenig's Lemma**

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The following variation of Koenig's lemma has proved useful in program semantics. Let $S$ and $T$ be arbitrary trees, possibly infinite. Each node of $S$ maps to an arbitrary set of nodes of $T$. Henceforth, whenever node $x$ of $S$ includes node $y$ of $T$ in its map we say $x$ *covers* $y$. The theorem below gives conditions under which every path of $T$ is covered by some path of $S$.

Formally, let $S$ and $T$ be two trees whose nodes we also designate by $S$ and $T$. Let *cover* be a binary relation over $S \times T$. Say that $x$ covers $y$ ($y$ is covered by $x$) if $(x, y) \in \mathit{cover}$, and for sets of nodes $X$, $Y$, where $X \subseteq S$ and $Y \subseteq T$, $X$ covers $Y$ ($Y$ is covered by $X$) if each node of $Y$ is covered by some node of $X$.

**Theorem** Given $S$, $T$ and *cover* as above, suppose:

C1. Each node of $T$ is covered by a non-empty finite set of nodes.

C2. If $x$ covers $y$ then the ancestors of $x$ in $S$ (that includes $x$) cover the ancestors of $y$.

Then every path of $T$ is covered by some path of $S$. □

Observe that a node of $S$ may cover zero, finite or infinite number of nodes, and there is no restriction on the degrees of nodes of $S$ and $T$. The theorem is of interest only when $T$ is infinite, because for finite $T$ every terminal node $y$ of $T$ is covered by some node of $S$ whose ancestors cover the path to $y$, from (C2).

**Proof of the Theorem** First, without loss in generality, add a new root $s$ to $S$, $t$ to $T$ and the pair $(s, t)$ to *cover*. Neither the hypotheses nor the conclusion are affected by this construction.

Let $p$ be a path in $T$ starting at $t$. Construct a tree $R$ from $S$ and $p$ as follows. The nodes of $R$ are $\{(x, y) \mid (x, y) \in \mathit{cover},\ y \in p\}$. Node $(x, y)$ is the parent of $(x', y')$, $y' \neq t$, where $y$ is the parent of $y'$ in $p$ and $x$ the ancestor of $x'$ closest to it in $S$ that covers $y$. Such an $x$ exists because of condition (C2). Node $x$ may possibly be $x'$. Every node in $R$ except $(s, t)$ has a parent. Observe:

1. $R$ is a tree with root $(s, t)$. Every node of $p$ is the second component of a distinct node of $R$. Hence, if $p$ is infinite so is $R$.
2. Every node of $R$ has finite degree: node $(x, y)$ of $R$ has children of the form $(x', y')$ where $y'$ is the unique child of $y$ in $p$. From (C1), $y'$ is covered by a finite number of nodes.
3. Apply Koenig's lemma in conjunction with items (1) and (2) to establish the existence of an infinite path $q$ in $R$. Let $q_1$ and $q_2$ be the sequences of first and second components, respectively, of $q$. By construction, $q_2 = p$. And $q_1$ corresponds to a path of $S$ that covers $p$, because $(x, y)$ is the parent of $(x', y')$ in $q$ where $x$ is an ancestor of $x'$ in $S$. The path corresponding to $q_1$ is finite if some node of $S$ appears infinitely often in $q_1$. □

Koenig's lemma is easily established from this theorem. Given an infinite tree with finite degree at each node, we have to show the existence of an infinite path. Let $S$ be the given tree and $T$ just a path whose nodes are numbered consecutively from the root with natural numbers. Let $x$ cover $n$ where $n$ is the level of $x$. Condition (C1) is met because every level has a finite non-zero number of nodes, and (C2) is easily seen to be met. So, there is a path of $S$ that covers $T$, and since each node of a path of $S$ has a different level, the path is infinite.

**A Stronger Version of the Theorem** Condition (C1) may be weakened as follows to obtain a stronger version of the theorem.

C1'. An infinite path in $T$ has an infinite number of nodes with finite coimage (coimage is the set of nodes that cover a specific node).

The proof can be modified as follows given (C1'). For an infinite path $p$ in $T$, construct a compressed path $p'$ retaining only the nodes that have finite coimage. Path $p'$ is infinite and it meets condition (C1) for the nodes in it. Use the given proof to create a path $q$ in $R$. As before, $q_1$ corresponds to a path in $S$ that may include additional nodes. This path in $S$, using (C2), covers all the nodes of $p$, including those that were removed during compression.