Identifying Lights with their Switches
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09/07/2012

Problem Description Given are N switches and N lights where each switch controls exactly one light and each light is controlled by exactly one switch. The wiring diagram is unavailable and the wiring itself is hidden. A *step* consists of selecting some number of switches and turning them on, and, presumably, noting the lights that come on as a result. It is required to determine which switch controls which light using a minimum number of steps. Clearly, N steps are sufficient; we show that exactly $\lceil \log_2 N \rceil$ steps are necessary and sufficient. Henceforth, $s = \lceil \log_2 N \rceil$.

A Procedure Number each switch arbitrarily by an s bit distinct binary string. We will assign exactly the same string to the light that this switch controls. Let step i turn on all the switches whose bit i is 1. Note all the lights that are turned on as a result, and assign bit position i for all the lights as follows: 1s to every light that turns on and 0s to the remaining lights. It is clear that every light has the correct bit in its i^{th} position. Applying the step for each bit position identifies all the lights with their corresponding switches.

A Lower Bound We show that there is no procedure that takes fewer number of steps. We assign a bit string to each light as a procedure goes through its sequence of steps: Initially, each light has an associated empty string; following a step, assign 1 to each light that comes on and 0s to all others as their next bit. If the number of steps is less than s, at least 2 lights are assigned the same bit string. Since the bit strings encode the results of all the steps, these two lights can not be distinguished by the given steps; for any pairing with two switches can be flipped without causing a difference in the outcomes of the steps.