Multiset Ordering: A Theorem of Manna and Dershowitz
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This note gives an elementary proof of a result, due to Manna and Dershowitz [1], on well-foundedness of an order relation over multisets (henceforth, called bags).

**Manna-Dershowitz Order** Let \((D, <)\) be a well-founded set. For finite bags \(X\) and \(Y\) over \(D\) define

\[
Y \prec X \equiv \langle (A \neq \emptyset \land (\forall y : y \in B : (\exists x : x \in A : y < x)) \rangle,
\]

where \(A = X - Y\) and \(B = Y - X\).

We may imagine that \(Y\) is constructed from \(X\) by removing the elements in \(A\) and adding the elements in \(B\). It is required that some element be removed \((A \neq \emptyset)\) and each element that is added be smaller than some removed element according to \(<\). We show that finite bags are well-founded under the \(\prec\) relation.

**Proof that \(\prec\) is a well-founded relation** Given a finite non-empty bag, a move removes a non-empty finite bag of elements from it and adds a finite bag of elements where for each added element \(y\) there is a removed element \(x\) such that \(y < x\). A game is a sequence of moves until the bag becomes empty; then no further move is possible and the game terminates.

To show that \(\prec\) is a well-founded relation, it is sufficient to prove that every game starting with a finite bag \(X\) terminates. To this end, construct a tree corresponding to a given game as follows. Imagine that there is an element \(\top\) in \(D\) such that for every other element \(x\) of \(D\), \(x < \top\). The bag initially contains just \(\top\), and the first move removes \(\top\) and adds the initial elements of \(X\).

1. The root of the tree is \(\top\).
2. There is a node corresponding to each element \(y\) that is added in a move.
3. Each node \(y\) has a parent \(x\) such that \(x < y\). This is possible because for every element that is added some “higher” element is removed. In case there is a choice for the higher element, any one is chosen arbitrarily.

The desired result follows from the following observations.

1. Each path in the tree is a decreasing chain in \(D\). Hence each path is finite, because \(D\) is well-founded.
2. Each node in the tree has finite degree. A move adds a finite number of elements as children of some terminal nodes.
3. Using Koenig’s lemma with the above two observations, the number of nodes in the tree is finite.
4. The number of moves in the game is at most the number of nodes in the tree. So the game is finite.
References