Multiset Ordering; A Theorem of Manna and Dershowitz
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This note gives an elementary proof of a result, due to Manna and Dershowitz [1], on well-foundedness of an order relation over multisets (henceforth, called *bags*).

Dershowitz-Manna order Let $(D, <)$ be a well-founded set. For finite bags X and Y over D define

$$Y \prec X \;\equiv\; \langle (A \neq \phi \;\wedge\; (\forall y : y \in B : \langle \exists x : x \in A : y < x \rangle) \rangle,$$
where $A = X - Y$ and $B = Y - X$.

We may imagine that Y is constructed from X by removing the elements in A and adding the elements in B. It is required that some element be removed ($A \neq \phi$) and each element that is added be smaller than some removed element according to $<$. Henceforth, $X \succ Y$ iff $Y \prec X$.

Example 1 Let D be the set of natural numbers with the standard order. Then $\{5, 5, 3, 2\} \succ \{5, 4, 4, 2\} \succ \{5, 4, 3, 3, 3, 1\} \succ \{5, 4\}$. Note that no element is added in the last step though some elements are removed.

Dershowitz-Manna order is a well-founded order We show that any decreasing chain of bags, $X_1 \succ X_2 \cdots$, is finite in length. To simplify the proof postulate a new element $\top$ outside D where $x < \top$ for all $x \in D$. Then $\{\top\} \succ S$ for any bag S over D. Let $X_0 = \{\top\}$ be the first bag in the chain. The first step removes $\top$ and adds all elements of X_0.

Assign a unique id to every element of every bag; call the element along with its id an *item.* If $y \notin X_i$ and $y \in X_{i+1}$ then y has been *added in step i*. So, there is some x *removed in step i*, i.e., $x \in X_i$ and $x \notin X_{i+1}$, such that $y \prec x$. Call x the *parent* of y; if there are several choices for a parent choose one arbitrarily.

Observe that every item in all X_i, $i > 0$, is added at some step because the initial bag, X_0, has none of the elements of D; so every item other than $\top$ has a parent. Construct a tree over all the items, including $\top$, using the parent relation defined above. The tree is connected and rooted at $\top$.

The desired result follows from the following observations.

1. Each path in the tree is a decreasing chain in D. Since D is well-founded each path is finite.

2. Each node in the tree has finite degree: All the children of a node x are added in the step in which x is removed. Only a finite number of items are added in each step. So, x has a finite number of children.

3. Using Kőnig's lemma, Lemma **??** (page **??**), with the above two observations the number of nodes in the tree is finite.

4. The number of steps in the chain is at most the number of nodes in the tree (items) because each step removes at least one item.

References

[1] Z. Manna and N. Dershowitz. Proving termination with multiset ordering. *Communications of the ACM*, 22(8):465–476, Aug 1979.