The Monty Hall game
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The *Monty Hall* game involves a player and a host. There are three closed doors one of which conceals a car and the other two conceal goats. The player chooses a door. The host, who knows what is behind each door, opens another door that conceals a goat. The player now has the option of retaining his original choice of door, or switching to the other closed door, and he then receives whatever that door conceals.

It is clear that the player has 1/3 chance of winning a car when he stays with his original choice, because his initial choice is random and the remaining actions of the host may be ignored. We give several short proofs that switching doors increases the probability of winning the car to 2/3.

Proof 1 After the player chooses a door, imagine that he splits himself into two *avatars* A and B, where A always stays with the initial choice and B always opts to switch. Between the two of them A and B always win; so the probability of winning is 1. Further, both A and B don't win because their choice of doors are different and there is just one car. Since A's probability remains 1/3, B's is 2/3.

Proof 2 Number the door that hides the car 0 and the others 1 and 2 (without telling the player of the numbering). The player chooses a random number r between 0 and 2. Suppose the player switches doors after the goat is revealed. If $r = 0$, he loses and otherwise he wins. Since r is chosen randomly with uniform probability, he wins 2/3 of the time.

Proof 3 The following proof was shown to me by Carroll Morgan, who heard it from someone else. Consider a different game G that permits the player to open any two doors and receive whatever is behind the doors. The probability of winning a car is 2/3 in this case. The player can simulate game G in the original Monty Hall game as follows. He chooses the door that he would have left closed in game G. Then the host opens one of the doors the player would have chosen in G. Next, given the chance to switch doors, the player switches and opens the other door of his choice in G.

Proof 4 We modify the original game as follows: after the host opens a door, the player flips a coin and chooses to switch iff the coin lands heads. Below, we abbreviate the events "the player chooses the correct door in his first guess" by *correct* and "the player chooses the incorrect door in his first guess" by *incorrect*. And the events "the coin lands tails" by *tails* and "the coin lands heads" by *heads*. Then the probability of the player winning a car is:

probability((*correct* and *tails*) or (*incorrect* and *heads*))
= {The two possibilities are disjoint.}

$$
\begin{array}{ll}
& \quad \text{probability}(\textit{correct} \text{ and } \textit{tails}) \\
& \; + \text{probability}(\textit{incorrect} \text{ and } \textit{heads}) \\
= & \{\text{The events in each case are independent.}\} \\
& \quad \text{probability}(\textit{correct}) \times \text{probability}(\textit{tails}) \\
& \; + \text{probability}(\textit{incorrect}) \times \text{the probability}(\textit{heads}) \\
= & \{\text{probability}(\textit{correct}) = 1/3 \text{ and probability}(\textit{incorrect}) = 2/3. \\
& \; \text{Let probability}(\textit{heads}) = p.\} \\
& 1/3 \times (1 - p) + 2/3 \times p = 1/3 + 1/3 \times p
\end{array}
$$

This expression is maximized by setting $p = 1$ so that the coin always lands heads, i.e., always choosing to switch.

A Final Note These proofs show that the host can choose *any* door that conceals a goat. *His choice does not have to be random in the case when he has a choice between two doors that both conceal goats.* For instance, he can always choose the lower-numbered door containing a goat, or use any other strategy.