The Muddy Children Puzzle
Jayadev Misra
10/8/98

Problem: There is a finite group of children where each child is *clean* or *dirty*. No child knows if it is clean or dirty, but it can see if every other child is clean or dirty. It is common knowledge that there is at least one dirty child.

In a *round*, (1) the children are asked: do you know if you are dirty, and (2) each of them responds with "NO", "YES, I am dirty", or "YES, I am clean". Responses are heard by all children. Rounds are repeated ad infinitum starting at round 0.

Prove that a child who sees n dirty children, $n \geq 0$, will answer YES in round n, but no earlier. (If there are n dirty children, $n > 0$, a dirty child sees $n-1$ dirty children and a clean child sees n dirty children; hence, all dirty children will answer YES in round $n-1$, but no earlier, and all clean children will answer YES in round n, but no earlier.)

Solution: Each solution employs a function , f, – called a *strategy* – that a child applies to its observations. Such a function has three arguments: n, C, D, where n is the number of dirty children it sees, C is the sequence of responses it has heard in the previous rounds from clean children and D is a sequence of responses it has heard in the previous rounds from dirty children. Sequence C is empty if a child hears no response from a clean child (when there is no clean child or the hearer is the only clean child). Similarly, D is empty if a child hears no response from a dirty child (when the hearer is the only dirty child).

From the symmetry, we can assume that all dirty children give the same answer in each round and so do all the clean children. We encode the responses "NO", "YES, I am dirty", "YES, I am clean" by integers $0, 1, 2$, respectively. Therefore, in $f(n, C, D)$, n is a natural number, C a sequence of $0, 2$ values and D a sequence of $0, 1$ values; the function value is from $\{0, 1, 2\}$.

Given a strategy f, let c_{in}, for $i \geq 0, n > 0$, be the response in round i of a clean child who sees n dirty children; and d_{in}, for $i \geq 0, n \geq 0$, be the response in round i of a dirty child who sees n dirty children. Note that c_{i0} is undefined since there is at least one dirty child. Let C_{in} be $\langle c_{0n}, .., \ c_{(i-1)n} \rangle$, the sequence of responses from clean children upto (but excludiung round i) when a clean child sees n dirty children. Similarly, D_{in} is defined.

We can define c_{in}, d_{in} recursively. If a clean child sees n dirty children, $n > 0$, it would have heard $c_{jn}, d_{j(n-1)}$ from the clean and dirty children, respectively, in round j, $0 \leq j < i$, i.e., $C_{in}, D_{i(n-1)}$. Similarly, a dirty child who sees n dirty children, $n > 0$, would have heard $c_{j(n+1)}, d_{jn}$ from the clean and dirty children, respectively, in round j, $0 \leq j < i$, i.e., $C_{i(n+1)}, D_{in}$. Therefore,

for $i \geq 0, \ n > 0$, $c_{in} = f(n, C_{in}, D_{i(n-1)})$

for $i \geq 0,\ n \geq 0,\ d_{in} = f(n, C_{i(n+1)}, D_{in})$

Validity: Each strategy satisfies

for $i \geq 0,\ n > 0,\ c_{in} \in \{0, 2\}$, and
for $i \geq 0,\ n \geq 0,\ d_{in} \in \{0, 1\}$.

Note: c_{in} may be different for the two cases where there is a single clean child – who sees no other clean child – and when there are more than one clean children. In the former case the answer is based on the responses heard from the dirty children only. Therefore, it is necessary to introduce
c_{imn} = the response of a clean child who sees m clean and n dirty children.
d_{imn} has analogous meaning.

In the following, it is not necessary to consider $m = 0,\ n = 0$, because we assume that there is more than one child. c_{imn} is not defined for $m > 0,\ n = 0$.

for $i \geq 0,\ m > 0,\ n > 0,$
$c_{imn} = f(m, n, C_{imn}, D_{i(m+1)(n-1)})$
$d_{imn} = f(m, n, C_{i(m-1)(n+1)}, D_{imn})$

for $i \geq 0,\ m = 0,\ n > 0,$
$c_{imn} = f(m, n, \langle\rangle, D_{i(m+1)(n-1)})$
$d_{imn} = f(m, n, \langle\rangle, D_{imn})$

for $i \geq 0,\ m > 0,\ n = 0,$
$d_{imn} = f(m, n, C_{i(m-1)(n+1)}, \langle\rangle)$

□

We propose a strategy ϕ below. Strategy ϕ is: a child who sees n dirty children answers YES in round i iff $i \geq n$. The child is dirty iff $n = 0$ or a dirty child answered NO in round numbered $n-1$. We show that (1) ϕ satisfies the Validity requirement, and (2) it is the "best" strategy.

Henceforth, elements of a sequence are indexed starting at 0; so, $C_{in}[m] = c_{mn}$. Let,

$$\phi(n; C, D) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } |D| < n \\ D[n-1] + 1 & \text{if } |D| \geq n \end{cases}$$

Note:: Strategy ϕ is defined when C is empty. Also, if $n = 0$ then the function value is independent of D.

Theorem 1:: Strategy ϕ satisfies,
for $i \geq 0,\ n > 0,\ c_{in} \in \{0, 2\}$, and $(c_{in} = 0) \equiv (i < n)$,

for $i \geq 0,\ n \geq 0,\ d_{in} \in \{0,1\}$, and $(d_{in} = 0) \equiv (i < n)$.

Proof::
For $n = 0$: $d_{i0} = \phi(0; -, -) = 1$. Hence, $d_{in} \in \{0,1\}$, and $(d_{in} = 0) \equiv (i < n)$.

For $n > 0$, we apply induction on i.

1. $i < n$: $c_{in} = \phi(n, C, D)$, where $|D| = i < n$. Hence, $c_{in} = 0$. Therefore, $c_{in} \in \{0,2\}$, and $(c_{in} = 0) \equiv (i < n)$. Similarly for d_{in}.

2. $i \geq n$:

$$\begin{array}{ll} & c_{in} \\ = & \{c_{in} = \phi(n, C_{in}, D_{i(n-1)}), \text{ where } |D_{i(n-1)}| = i \geq n\} \\ & D_{i(n-1)}[n-1] + 1 \\ = & \{D_{i(n-1)}[n-1] = d_{(n-1)(n-1)}. \text{ By induction, } d_{(n-1)(n-1)} = 1\} \\ & 2 \end{array}$$

Therefore, $c_{in} \in \{0,2\}$, and $(c_{in} = 0) \equiv (i < n)$.

$$\begin{array}{ll} & d_{in} \\ = & \{c_{in} = \phi(n, C_{i(n+1)}, D_{in}), \text{ where } |D_{in}| = i \geq n\} \\ & D_{in}[n-1] + 1 \\ = & \{D_{in}[n-1] = d_{(n-1)n}. \text{ By induction, } d_{(n-1)n} = 0\} \\ & 1 \end{array}$$

Therefore, $d_{in} \in \{0,1\}$, and $(d_{in} = 0) \equiv (i < n)$. $\square$

Let $f.c_{in}, f.d_{in}$ denote the values of c_{in}, d_{in}, computed using strategy f. Also, $f.C_{in}, f.D_{in}$ denote the corresponding sequences computed using f.

Definition: For strategies $f, g,\ f \leq g$, if
$\langle \forall i, n : i \geq 0, n > 0 : f.c_{in} \leq g.c_{in} \rangle$, and
$\langle \forall i, n : i \geq 0, n \geq 0 : f.d_{in} \leq g.d_{in} \rangle$.

In this case, strategy g is at least "as good as" strategy f because it yields YES whenever f yields YES. We say g is *stronger* than f.

Theorem 2:: For all $f,\ f \leq \phi$.

Proof:: Let f be any strategy. We show that $\phi.c_{in} = 0 \Rightarrow\ f.c_{in} = 0$, for all $i, n : i \geq 0, n > 0$. A similar proof applies for d_{in}. Proof is by induction on i. For $i = 0, n > 0$,

$$\begin{array}{ll} & f.c_{in} = f(n, \langle\rangle, \langle\rangle),\ f.d_{in} = f(n, \langle\rangle, \langle\rangle) \\ \Rightarrow & \{\text{predicate calculus}\} \\ & f.c_{in} = f.d_{in} \\ \Rightarrow & \{\text{From the Validity requirement, } f.c_{in} \in \{0,2\},\ f.d_{in} \in \{0,1\}\} \\ & f.c_{in} = 0 \end{array}$$

For $i > 0, n > 0$: we use $\overline{0}$ as a sequence of length i consisting of zeroes .

$$
\begin{array}{ll}
 & \phi.c_{in} = 0 \\
\Rightarrow & \{\text{From theorem 1, } \phi.c_{in} = 0 \Rightarrow i < n. \text{ Apply definition of } \phi\} \\
 & \phi.C_{in}, \phi.D_{i(n-1)}, \phi.C_{i(n+1)}, \phi.D_{in} = \overline{0}, \overline{0}, \overline{0}, \overline{0} \\
\Rightarrow & \{\text{Induction Hypothesis}\} \\
 & f.C_{in}, f.D_{i(n-1)}, f.C_{i(n+1)}, f.D_{in} = \overline{0}, \overline{0}, \overline{0}, \overline{0} \\
\Rightarrow & \{f \text{ is a strategy}\} \\
 & f.c_{in} = f(n, f.C_{in}, f.D_{i(n-1)}) \ \wedge \ f.d_{in} = f(n, f.C_{i(n+1)}, f.D_{in}) \\
 & \ \wedge \ f.C_{in}, f.D_{i(n-1)} = f.C_{i(n+1)}, f.D_{in} \\
\Rightarrow & \{f.c_{in},\ f.d_{in} \text{ have the same arguments}\} \\
 & f.c_{in} = f.d_{in} \\
\Rightarrow & \{\text{From the Validity requirement, } f.c_{in} \in \{0,2\},\ f.d_{in} \in \{0,1\}\} \\
 & f.c_{in} = 0
\end{array}
$$

□

Note::

1. In this proof, we have assumed that there is a clean child; so, the list C_{in} is non-empty for $i > 0$. A similar, simpler, proof applies if there is no clean child, because the definition of ϕ does not rely on C.

2. If the requirement "there is at least one dirty child" is dropped then it is easy to show that, for all i, n, c_{in}, d_{in} are both 0. That is, the children can never answer YES. Redefine ϕ to yield 0 in all cases, and prove that ϕ dominates all functions.