Guessing with N Hat colors

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Problem Description The following problem was communicated to me by Mike Starbird. There are N persons each of whom has a hat on his head. There are N possible hat colors. Not every color appears on someone's head. Every one can see the colors of all others' hats, but not his own. Each person guesses the color of his hat (writing it down on a piece if paper, say). The group devises a protocol, before the hats are put on their heads, so that at least one person guesses his hat color correctly. There is no communication among the group members after they get their hats.

Protocol Number the persons from 0 through $N-1$ and also the colors. Let h_j be the color of j's hat and H the sum of colors of all the hats. Now, j can see all the hats except his own; so, he can only compute $H_j = H - h_j$.

If j can guess the value of H correctly then he can compute $h_j = H - H_j$. In fact, if he can guess (H mod N) correctly, he can also do the same, for

$$\begin{array}{ll} & (H \bmod N - H_j) \bmod N \\ = & \{\text{modulo arithmetic}\} \\ & (H - H_j) \bmod N \\ = & \{h_j = H - H_j\} \\ & h_j \bmod N \\ = & \{0 \leq h_j < N\} \\ & h_j \end{array}$$

Now, (H mod N) has N possible values, from 0 through $N-1$. Let j guess (H mod N) to have value j, and therefore guess his hat color to be $(j - H_j)$ mod N. Since every person has a different index, ranging over 0 through $N-1$, some one guesses (H mod N) correctly and, therefore, guesses his own hat color correctly.

A Derived Problem Given the same problem but with only t colors for hats, devise a protocol such that $\lfloor N/t \rfloor$ persons guess correctly. This is particularly interesting with $t = 2$.

The protocol is a small generalization of the given one. Number the persons from 0 through $t-1$ in a round-robin fashion so that every number from 0 through $t-1$ is assigned to a group of at least $\lfloor N/t \rfloor$ persons. Then j guesses his hat color to be $(j - H_j)$ mod t, based on his guess that H mod $t = j$. Some group guesses (H mod t) correctly, and all members of that group correcdtly guess their hat colrs. Note that the group size is at least $\lfloor N/t \rfloor$.

For $t = 2$, the problem can be stated and solved more easily. Each person has a white or a black hat; encode white by 0 and black by 1. Then (H mod 2) is the parity of all hats. Let $\lfloor N/2 \rfloor$ people guess (H mod 2) to be 0, and thus guess their own hat colors to be the parity of all hats they see. The remaining

$\lceil N/2 \rceil$ persons guess ($H \bmod 2$) to be 1, thus guess their own hat color to be the opposite of the parity of all hats they see.