A Note on EWD 1312
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This note is prompted by EWD 1312.

Integers h and k are *disjoint* provided in their binary representations no position has 1s in both h and k. (In comparing two integers, append leading zeroes to the binary representation of the smaller number to make their lengths identical.) Dijkstra proved that for positive h and k, among (h,k), $(h,k-1)$, and $(h-1,k)$, an even number of pairs are disjoint. In this note we prove this result using some elementary algebraic properties of disjointness. Henceforth, we view a nonnegative integer as a binary string.

Let $\overline{x}$ be the binary string obtained by complementing each bit in x. Note that $\overline{\overline{x}}$ is x. For strings x and y of equal length, write $x \sqsupseteq y$ to mean that each bit of x is at least the corresponding bit of y; note that for empty strings x and y, $x \sqsupseteq y$.

Observation:

1. $x \sqsupseteq y \;\equiv\; \overline{y} \sqsupseteq \overline{x}$.
2. (x, y) are disjoint $\equiv\; \overline{x} \sqsupseteq y$, (or, equivalently, $\overline{y} \sqsupseteq x$).
3. $(x \sqsupseteq y \;\wedge\; \overline{x} \sqsupseteq y) \;\equiv\; y = 0$.

Positive integers h and k have some 1 bits in their binary representations. Let $h = pq$ where q's leading bit is 1 and its other bits are all zeroes (pq denotes the concatenation of the strings p and q). Since the problem is symmetric in h and k, assume that the number of trailing zeroes in h do not exceed those in k. Let $k = rs$ where the lengths of p and r are equal (then, so are the lengths of q and s). Now, $h - 1 = p\overline{q}$ and $k - 1 = t\overline{s}$, for some t. Also, considering the two cases where h, k have different and equal numbers of trailing zeroes, we get (P1), below. (P2) follows from the definition of q.

P1. $(s = 0) \;\vee\; (t = r \;\wedge\; s = q)$.

P2. $q \neq 0$.

The three propositions under consideration are, using observation 2:

$\alpha :\; \overline{pq} \sqsupseteq rs$, i.e., (h, k) are disjoint,
$\beta :\; \overline{p}q \sqsupseteq rs$, i.e., $(h - 1, k)$ are disjoint,
$\gamma :\; \overline{pq} \sqsupseteq t\overline{s}$, i.e., $(h, k - 1)$ are disjoint.

We compute the values of α, β, and γ.

Lemma 1: $\alpha \;=\; (\overline{p} \sqsupseteq r) \;\wedge\; (s = 0)$

Proof:

$$
\begin{array}{ll}
 & \overline{pq} \sqsupseteq rs \\
\equiv & \{\text{definition of } \sqsupseteq\} \\
 & \overline{p} \sqsupseteq r \ \wedge \ \overline{q} \sqsupseteq s \\
\equiv & \{q \sqsupseteq s, \text{ from P1. Then, } \overline{q} \sqsupseteq s \ \equiv \ s = 0, \text{ from observation 3.}\} \\
 & \overline{p} \sqsupseteq r \ \wedge \ s = 0
\end{array}
$$

Lemma 2: $\beta \ = \ (\overline{p} \sqsupseteq r)$
Proof:

$$
\begin{array}{ll}
 & \overline{p}q \sqsupseteq rs \\
\equiv & \{q \sqsupseteq s, \text{ from P1}\} \\
 & \overline{p} \sqsupseteq r
\end{array}
$$

Lemma 3: $\gamma \ = \ (\overline{p} \sqsupseteq r) \ \wedge \ s \neq 0$
Proof:

$$
\begin{array}{ll}
 & \overline{pq} \sqsupseteq t\overline{s} \\
\equiv & \{\text{definition of } \sqsupseteq\} \\
 & \overline{p} \sqsupseteq t \ \wedge \ \overline{q} \sqsupseteq \overline{s} \\
\equiv & \{\overline{q} \sqsupseteq \overline{s} \ \equiv \ s \sqsupseteq q, \text{ from observation 1.} \\
 & \quad \text{From P1 and P2, } (t = r \ \wedge \ s = q) \text{ and } q \neq 0.\} \\
 & \overline{p} \sqsupseteq r \ \wedge \ s = q \\
\equiv & \{\text{From P1 and P2}\} \\
 & \overline{p} \sqsupseteq r \ \wedge \ s \neq 0
\end{array}
$$

Theorem: An even number of propositions from $\{\alpha, \beta, \gamma\}$ hold.
Proof: We show that $\alpha \ \equiv \ \beta \ \equiv \ \gamma$ is *false*.

$$
\begin{array}{ll}
 & \alpha \ \equiv \ \beta \ \equiv \ \gamma \\
\equiv & \{\text{Predicate Calculus}\} \\
 & \beta \ \equiv \ (\alpha \ \equiv \ \gamma) \\
\equiv & \{\text{from Lemmas 1,3}\} \\
 & \beta \ \equiv \ \langle(\overline{p} \sqsupseteq r) \ \wedge \ (s = 0)\rangle \equiv \ \langle(\overline{p} \sqsupseteq r) \ \wedge \ s \neq 0\rangle \\
\equiv & \{\text{Predicate Calculus}\} \\
 & \beta \ \equiv \ \neg(\overline{p} \sqsupseteq r) \\
\equiv & \{\text{Lemma 2}\} \\
 & (\overline{p} \sqsupseteq r) \ \equiv \ \neg(\overline{p} \sqsupseteq r) \\
\equiv & \{\text{Predicate Calculus}\} \\
 & \textit{false}
\end{array}
$$