Pairing Integers so that their sums are primes

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The following problem was shown to me by Gérard Huet.

Problem: For every even positive integer n, pair the integers up to n so that the sum of each pair is prime.

For $n = 2$, the pairing is $(\{1, 2)\}$. For $n = 4$, a pairing is $(\{1, 2), (3, 4)\}$, and another is $(\{1, 4), (2, 3)\}$.

Proof: Henceforth, all integers are positive. We make use of the following theorem, postulated by Bertrand (in 1845) and proved by Chebyshev (in 1850): For every integer m, $m \geq 2$, there is a prime p, $m < p < 2 \times p$. We use a special case of this result. Noting that p is odd: for every even integer n, there is an odd integer i, $i < n$, such that $n + i$ is prime.

We prove the required result using induction on n.

- $n = 2$: the pairing is $(\{1, 2)\}$.

- $n > 2$: Let i be as given by Bertrand's theorem. First, we pair the integers between i and n inclusive. Pair k with $n + i - k$. Their sum, $n + i$, is prime. Note that the pairing rule is valid because: (1) pairing is symmetric: $n + i - k$ is paired with $n + i - (n + i - k)$, i.e., k, and (2) members of a pair are distinct because their sum, $n + i$, is odd.

Next, we pair the integers up to $i - 1$. If $i = 1$, the pairing is vacuous. Otherwise, $i - 1$ is even (because i is odd) and $i - 1 < n$. From the induction hypothesis, there is a pairing up to $i - 1$.