Pairing Integers so that their sums are primes
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The following problem was shown to me by Gérard Huet.

**Problem:** For every even positive integer \( n \), pair the integers up to \( n \) so that the sum of each pair is prime.

For \( n = 2 \), the pairing is \((1, 2)\). For \( n = 4 \), a pairing is \((1, 2), (3, 4)\), and another is \((1, 4), (2, 3)\).

**Proof:** Henceforth, all integers are positive. We make use of the following theorem, postulated by Bertrand (in 1845) and proved by Chebyshev (in 1850): For every integer \( m, m \geq 2 \), there is a prime \( p, m < p < 2 \times p \). We use a special case of this result. Noting that \( p \) is odd: for every even integer \( n \), there is an odd integer \( i, i < n \), such that \( n + i \) is prime.

We prove the required result using induction on \( n \).

- \( n = 2 \): the pairing is \((1, 2)\).

- \( n > 2 \): Let \( i \) be as given by Bertrand’s theorem. First, we pair the integers between \( i \) and \( n \) inclusive. Pair \( k \) with \( n + i - k \). Their sum, \( n + i \), is prime. Note that the pairing rule is valid because: (1) pairing is symmetric: \( n + i - k \) is paired with \( n + i - (n + i - k) \), i.e., \( k \), and (2) members of a pair are distinct because their sum, \( n + i \), is odd.

Next, we pair the integers up to \( i - 1 \). If \( i = 1 \), the pairing is vacuous. Otherwise, \( i - 1 \) is even (because \( i \) is odd) and \( i - 1 < n \). From the induction hypothesis, there is a pairing up to \( i - 1 \).