Parities of Binomial Coefficients
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5/16/01

In this note, variables k, n are integers and $\binom{n}{k}$ is a binomial coefficient. We write $\left\langle {n \atop k} \right\rangle$ for the lowest bit, i.e., parity, of $\binom{n}{k}$. We derive a result about parity bits.

Theorem: For any natural number t, and integers k and n,

$$\left\langle {n+2^t \atop k} \right\rangle = \left\langle {n \atop k} \right\rangle \oplus \left\langle {n \atop k-2^t} \right\rangle$$

Proof: Proof is by induction on t.

Case $t = 0$: We have to show

$$\left\langle {n+1 \atop k} \right\rangle = \left\langle {n \atop k} \right\rangle \oplus \left\langle {n \atop k-1} \right\rangle$$

We exploit the following identity over binomial coefficients.

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

The lowest bit of the lhs is 0 iff the terms in the rhs have identical lowest bits. Hence the result.

Case $t+1$, $t \geq 0$:

$$\left\langle {n+2^{t+1} \atop k} \right\rangle$$

$=$ {rewriting}

$$\left\langle {(n+2^t)+2^t \atop k} \right\rangle$$

$=$ {induction}

$$\left\langle {n+2^t \atop k} \right\rangle \oplus \left\langle {n+2^t \atop k-2^t} \right\rangle$$

$=$ {induction applied to both terms}

$$\left\langle {n \atop k} \right\rangle \oplus \left\langle {n \atop k-2^t} \right\rangle \oplus \left\langle {n \atop k-2^t} \right\rangle \oplus \left\langle {n \atop k-2^{t+1}} \right\rangle$$

$=$ {property of $\oplus$}

$$\left\langle {n \atop k} \right\rangle \oplus \left\langle {n \atop k-2^{t+1}} \right\rangle$$