A puzzle on infinite sequences: An application of Koenig's Lemma

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4/12/96

Problem: Define a *word* to be any non-empty finite sequence of symbols. Each word is either *good* or *bad*. Given an infinite sequence of symbols, show that beyond some point, the sequence can be broken into words that are all good or that are all bad.

I saw this problem in a book by Dennis Sasha in the chapter on Leonid Levin.

Solution: The proof strategy is to construct a tree out of the given sequence, S. The root of the tree is a special symbol; the remaining nodes are labelled with natural numbers. The node labelled 0 has root as its father; the node labelled j has a father i if the word $S_i, S_{i+1}, ..., S_{j-1}$ is good, and i is the largest value below j having this property. If there is no such i, then the father of j is the root.

According to Keonig's lemma there is either an infinite path – in that case the entire sequence S can be broken up into good words – or a node with infinite degree. In the latter case, let the children of the node have labels $k_0, k_1, ...$ Each word starting at k_t and ending at $k_{t+1} - 1$ is a bad word, for all t.