Reducing Satisfiability to Quadratic Programming
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We show that the boolean satisfiability problem can be reducing to the quadratic programming problem.

1 Reduction

Let x, y, z be the variables. Denote their negations by $\overline{x}, \overline{y}, \overline{z}$. For a disjunct of the form $\neg x \vee y \vee z$ create an equation

$$\overline{x} + y + z \geq 1$$

For each variable x, create inequalities

$$x \geq 0, \; \overline{x} \geq 0$$

The objective function is

$$\min \textstyle\sum_x (x\overline{x})$$

2 Properties

P1: The objective function value is non-negative: because each term is non-negative.

P2: There is a feasible solution: Set each variable to 1. Another feasible solution is to set x to $1/k$ where k is the smallest number of terms in any equation in which x appears.

P3: If the boolean formula is satisfiable then the objective function value is 0.
Proof: In satisfying the boolean formula, for each variable that is *true*/*false* set its value in the quadratic programming problem to 1/0. Each inequality is satisfied and $x\overline{x} = 0$. Therefore, using P1, the objective function value is 0.

P4: If function value is 0 then the boolean formula is satisfiable.
Proof: Since each term in the objective function is non-negative, the value of the function is 0 only if each term $x\overline{x} = 0$. Then one of x and $\overline{x}$ is 0. Call the literal that is 0 *lo* and the other *hi*; if both literals are 0 then choose *lo*, *hi* arbitrarily. We claim that setting each *hi* literal to *true* and *lo* literal to *false* solves the boolean satisfiability problem. Consider an x that is *hi*. Every inequality in which x appears is satisfied because $x = 1$. Every inequality in which x does not appear is unaffected.
Note: It may be preferable to add $x + \overline{x} \leq 1$ to bound the polyhedron.