A Proof of Infinite Ramsey Theorem

Jayadev Misra

07/31/2012

The following theorem is due to Ramsey.

Theorem: Consider a completely connected graph over an infinite number of nodes where each edge is colored using one of a finite set of colors. Then the graph contains an infinite subset of nodes in which all edges between nodes in the subset have the same color.

Proof: Let V_0 be the set of nodes of the graph. Let v_0 be an arbitrary node of V_0. Since v_0 has an infinite number of edges incident on it and each edge has a color drawn from a finite set, some color, c_0, is the color of infinitely many of these edges. Let V_1 be the set of neighbors of v_0 on which these edges are incident, i.e. for every node x in V_1, the edge (v_0, x) is colored c_0. Observe that $V_1 \subset V_0$. The step is shown schematically in Figure 1.

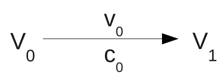

Figure 1: A single step of Construction

Now, V_1 is an infinite set and the same construction can be applied to it to obtain v_1, c_1 and an infinite set of V_2. Applying the construction repeatedly, we obtain an infinite sequence of infinite sets V_0, V_1, $\cdots$, V_i, $\cdots$, and the corresponding nodes and colors, v_i and c_i, as shown schematically in Figure 2.

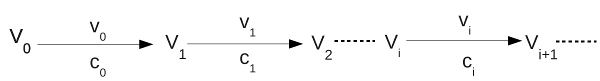

Figure 2: Repeated Construction Steps

Note that for all i, $i > 0$,

1. $v_i \in V_i$
2. $V_{i+1} \subset V_i$
3. (v_i, x) is colored c_i for every x in V_{i+1}.

We claim that for any i and j, where $i < j$, the edge (v_i, v_j) is colored c_i. From (1), $v_j \in V_j$, and from (2) $V_j \subset V_{j-1} \subseteq \cdots V_{i+1}$. So, $v_j \in V_{i+1}$. From (3), (v_i, v_j) is colored c_i.

The number of colors being finite some color, c, occurs infinitely often among the colors c_0, c_1, $\cdots$, c_i, $\cdots$. Let $R = \{v_i | \ c_i = c\}$. We show that R satisfies the requirement of the theorem. First, R is infinite by construction. Second, any edge (v_i, v_j), where both nodes are in R, is colored c_i, hence c, according to the above claim.

Remark: The proof is easily generalized to hypergraphs where an edge is a subset of nodes of some fixed size.