A cat, mouse puzzle

Jayadev Misra
Dept. of Computer Science
The University of Texas
Austin, Texas, 78712

January 19, 2023

This note gives a strategy for a cat to catch a mouse when the mouse moves among a finite number of caves in a non-deterministic fashion. It is a nice example of sequential program verification.

Problem statement There are a finite number of caves numbered from 0 through N. A mouse spends a day in one of the caves and moves to an adjacent cave next morning. Specifically, given m as the cave number in which the mouse spends a day, it moves next morning to some cave in set $next(m)$ where

$$next(m) = \begin{cases} \{1\} & \text{if} \quad m = 0 \\ \{N-1\} & \text{if} \quad m = N \\ \{m-1, m+1\} & \text{otherwise} \end{cases}$$

A cat knows the movement protocol of the mouse, but does not know its initial position nor, of course, which non-deterministic choice it will make each day. The cat can visit any one cave each day; its position is given by c. Devise a strategy for the cat to catch the mouse, i.e. ensure that eventually $m = c$.

Cat's strategy The cat makes two linear searches (rounds) over the caves, from N down to 0, and from 0 to N; between the rounds it spends two consecutive days in cave 0 if it has not succeeded until then. I show that eventually $m = c$. The strategy is given by the program in Figure 1.

Correctness There is the obvious invariant that the cat and mouse are in some caves: $I :: \; 0 \le m, \; c \le N$. Since m can increase or decrease by at most 1, we can assert, $A :: \; \{ \; m = k \; \} \; \; m :\in next(m) \; \; \{ \; k-1 \le m \le k+1 \; \}$.

The main insight in the correctness proof is that if the mouse stays in an odd-numbered cave one day then it is in an even cave the next day, and vice versa. The same observation applies to the cat because it always moves to an adjacent cave. Therefore, if the cat and mouse occupy caves of the same parity at the start of a round, they will continue to occupy caves of the same parity each day.

Write $m \sim c$ to denote that m and c have the same parity. I show below that the cat catches the mouse if it starts a round, say at $c = N$, with the same parity as the mouse. The invariant of the search loop is $I,\ m \sim c,\ m \leq c$, as shown by the annotation. The loop terminates because c decreases in each iteration, and it is bounded below by 0, from I. On termination of the loop $m = c$.

```
{ I, m ~ c, c = N }
{ I, m ~ c, m ≤ c }
while  m ≠ c  do
{ I, m ~ c, m < c. Since m ~ c, m < c implies m + 1 < c }
  m :∈ next(m); c := c − 1
{ I, m ~ c. From assertion A, m ≤ c }
enddo
```

Full program The complete program with both rounds is shown in Figure 1. The first round succeeds for the cat if $m \sim c$ holds during that round. Otherwise, at the end of that round $m \not\sim c$ and $c = 0$. So, by staying in cave 0 for another day, the cat ensures that $m \sim c$ and the second round will succeed.

For the formal proof, use $(m \sim c) \Rightarrow (m \leq c)$ as the invariant of the first loop, and $(m \sim c) \wedge (m \geq c)$ as the invariant of the second loop.

Cat, mouse dance

```
{ 0 ≤ m ≤ N }
{ Initialize }
c := N;

{ The first round of search }
while  m ≠ c ∧ c ≠ 0  do
  m :∈ next(m); c := c − 1
enddo ;

{ Now either m = c or c = 0 }
if  m ≠ c  then  { c = 0 }
{ Start the second round of search. }
  m :∈ next(m);
  while  m ≠ c  do
    m :∈ next(m); c := c + 1
  enddo
endif
{ m = c }
```

Figure 1: Cat is guaranteed to catch the mouse