A Theorem About Dynamic Acyclic Graphs

Notes on UNITY: 02-88

Jayadev Misra*
Department of Computer Sciences
The University of Texas at Austin
Austin, Texas 78712
(512) 471-9547
misra@cs.utexas.edu

September 13, 1988

1 The Problem

Given is a finite directed acyclic graph. A *top* vertex has no incoming edge; a *bottom* vertex has no outgoing edge. The graph is changed according to the following operation: Pick an arbitrary vertex; if it is a top vertex reverse the directions of all edges incident on it (thereby converting it into a bottom vertex). This operation is repeated forever with the constraint that every vertex is picked infinitely often. It is required to show that any vertex eventually becomes a bottom vertex.

This is a special case of the dining philosophers solution given in Chapter 12 of [1]. The dining philosophers has, additionally, a state for each vertex: thinking, hungry, or eating. Every eating vertex is a bottom vertex. The rule for transformation of the graph is this: A hungry vertex that has no incoming edge from a hungry vertex either becomes eating (and points all incident edges inward to become a bottom vertex) or gets an incoming edge from a hungry vertex (by some thinking vertex becoming hungry). It was proven that every hungry vertex eventually becomes eating.

The purpose of this note is to study the proof of the simpler problem (stated in the first paragraph) which abstracts the essence of the progress proof for the dining philosophers solution in [1].

2 A Formal Statement of the Problem

Let u, v, w denote arbitrary vertices, $u\ E\ v$ denotes that there is a directed edge from u to v and $u\ R\ v$ denotes that there is a directed path from u to v. Formal definition of $u\ R\ v$ is as follows:

$$\begin{array}{lll} u\ R^1\ v & \equiv u\ E\ v & \\ u\ R^{n+1}\ v & \equiv \langle \exists\ w\ ::\ u\ E\ w\ \wedge\ w\ R^n\ v \rangle & ,\ n \geq 1 \\ u\ R\ v & \equiv \langle \exists\ n\ :\ n \geq 1\ ::\ u\ R^n\ v \rangle & \end{array}$$

We write $u.\top, u.\perp$ to denote, respectively, that u is a top or bottom vertex. Formally,

*This work was partially supported by ONR Contracts N00014-87-K-0510 and N00014-86-K-0763 and by a grant from the John Simon Guggenheim Foundation.

$u.\top \;\equiv\; \langle \forall\, v \;::\; \neg v \; E \; u \rangle$
$u.\bot \equiv\; \langle \forall\, v \;::\; \neg u \; E \; v \rangle$

Note: A disconnected vertex is both a top and a bottom vertex. $\bigtriangledown$

Convention: Throughout this note $n \geq 1$. $\bigtriangledown$

The following program describes the way the graph is changed.

Program *change*

initially $\langle [\!] \; u \;::\; u \; R \; u = \mathit{false} \rangle$ {graph is initially acyclic}
assign
$\langle [\!] \; u \;::$
$\langle \| \; v \;::\; u \; E \; v, \; v \; E \; u \;:=\; \mathit{false}, \; u \; E \; v \quad \text{if} \quad u.\top \rangle$
$\rangle$
end {*change*}

It is required to show that for any u,

$\mathit{true} \;\mapsto\; u.\bot$

3 Basic Facts

We only need the following properties of program *change*.

initially $\neg u \; R \; u$ (P0)
$\neg u \; E \; v \;\mathit{unless}\; \neg u.\bot \;\wedge\; v.\bot$ (P1)
$u.\top \;\mapsto\; u.\bot$ (P2)

Properties (P0,P1) can be proven directly from the program text. Property (P2) is proven by showing $u.\top \;\mathit{ensures}\; u.\bot$, as follows.

Proof of $u.\top \;\mathit{unless}\; u.\bot$:
$\neg v \; E \; u \;\mathit{unless}\; u.\bot$
, interchanging u, v in (P1) and weakening its right hand side
$\langle \forall\, v \;::\; \neg v \; E \; u \rangle \;\mathit{unless}\; u.\bot$
, simple disjunction
$u.\top \;\mathit{unless}\; u.\bot$
, definition of $u.\top$ $\bigtriangledown$

Next observe that there is a statement that establishes $u.\bot$ as a postcondition given $u.\top$ as a precondition.

$$\{u.\top\} \;\; \langle \| \; v \;::\; u \; E \; v, \; v \; E \; u \;:=\; \mathit{false}, \; u \; E \; v \quad \text{if} \quad u.\top \rangle \;\; \{u.\bot\}$$

The proof of the following theorem is straightforward from the definitions.

Theorem 0: $u \; R^n \; v \;\Rightarrow\; \neg u.\bot$
Proof (sketch): Show, using induction on n, that
$u \; R^n \; v \;\Rightarrow\; \langle \exists\, w \;::\; u \; E \; w \rangle$
From definition of $u.\bot$, $\langle \exists\, w \;::\; u \; E \; w \rangle \;\Rightarrow\; \neg u.\bot$ $\bigtriangledown$

Corollary 0: $u \; R \; v \;\Rightarrow\; \neg u.\bot$ $\bigtriangledown$

4 The Graph Remains Acyclic

We show that no path is created from one vertex to another unless the former becomes nonbottom and the latter bottom.

Theorem 1: $\neg u\ R^n\ v\ \textit{unless}\ \neg u.\bot\ \wedge\ v.\bot$
Proof: The proof is by induction on n.

$n=1$: $\neg u\ E\ v\ \textit{unless}\ \neg u.\bot\ \wedge\ v.\bot$, from (P1)

$n+1$: $\neg u\ R^{n+1}\ v = \langle\forall\ w\ ::\ \neg u\ E\ w\ \vee\ \neg w\ R^n\ v\rangle$, from the definition of R^{n+1}

$\neg u\ E\ w\ \textit{unless}\ \neg u.\bot\ \wedge\ w.\bot$, from (P1)

$\neg w\ R^n\ v\ \textit{unless}\ \neg w.\bot\ \wedge\ v.\bot$, induction hypothesis

$\neg u\ E\ w\ \vee\ \neg w\ R^n\ v\ \textit{unless}$
$(u\ E\ w\ \wedge\ \neg w.\bot\ \wedge\ v.\bot)$
$\vee\ (w\ R^n\ v\ \wedge\ \neg u.\bot\ \wedge\ w.\bot)$
$\vee\ (\neg u.\bot\ \wedge\ w.\bot\ \wedge\ \neg w.\bot\ \wedge\ v.\bot)$, disjunction of the above two

The first disjunct in the right side of the above may be weakened to $\neg u.\bot$ (because $u\ E\ w \Rightarrow \neg u.\bot$, from Theorem 0) $\wedge\ v.\bot$; the second disjunct is *false* because $w\ R^n\ v \Rightarrow \neg w.\bot$ (from Theorem 0); the third disjunct is *false*. Hence we get

$\neg u\ E\ w\ \vee\ \neg w\ R^n\ v\ \textit{unless}\ \ \neg u.\bot\ \wedge\ v.\bot$
$\neg u\ R^{n+1}\ v\ \qquad \textit{unless}\ \ \neg u.\bot\ \wedge\ v.\bot$, simple conjunction of the above over all w $\bigtriangledown$

Corollary 1: $\neg u\ R\ v\ \textit{unless}\ \neg u.\bot\ \wedge\ v.\bot$
Proof: Take simple conjunction of Theorem 1 over all n. $\bigtriangledown$

The following corollary says that the graph is always acyclic.

Corollary 2: $\neg u\ R\ u$ is invariant
initially $\neg u\ R\ u$, from (P0)
$\neg u\ R\ u$ is stable , setting v to u in Corollary 1. $\bigtriangledown$

Let $u.a$ denote the ancestors of, i.e., the set of vertices that have paths to, u. Formally,

$$v \in u.a\ \equiv\ v\ R\ u$$

Let A dnote any constant set of vertices.

Corollary 3: $u.a = A\ \textit{unless}\ u.a \subset A\ \vee\ u.\bot$
Proof:

$\neg v\ R\ u\ \textit{unless}\ u.\bot$
, from Corollary 1 interchanging u, v and weakening the right hand side

$v \notin u.a\ \textit{unless}\ u.\bot$
, from the above using the definition of $u.a$

$\langle\forall\ v\ :\ v \notin A\ ::\ v \notin u.a\rangle\ \textit{unless}\ u.\bot$
, simple conjunction of the above over all $v \notin A$

$u.a \subseteq A\ \textit{unless}\ u.\bot$
, the above left hand side is $\langle\forall\ v\ ::\ v \notin\ A\ \Rightarrow\ v \notin u.a\rangle$, i.e., $u.a \subseteq A$

$u.a = A\ \textit{unless}\ u.a \neq A$
, antireflexivity of *unless*

$u.a = A\ \textit{unless}\ u.a \subset A\ \vee\ u.\bot$
, conjunction of the above two and weakening the right hand side $\bigtriangledown$

5 The Progress Proof

Theorem 2: $\mathit{true} \;\mapsto\; u.\bot$

Proof:

$u.a = A \;\mathit{unless}\; u.a \subset A \;\vee\; u.\bot$

, Corollary 3

$v \in A$ is stable

, A is a constant set

$u.a = A \;\wedge\; v \in A \;\mathit{unless}\; u.a \subset A \;\vee\; u.\bot$

, conjunction of the above two and weakening the right hand side

$v.\top \;\mapsto\; v.\bot$

, property (P2)

$u.a = A \;\wedge\; v \in A \;\wedge\; v.\top \;\mapsto\; (u.a = A \wedge v \in A \wedge v.\bot) \;\vee\; u.a \subset A \;\vee\; u.\bot$

, PSP rule on the above two

$u.a = A \;\wedge\; v \in A \;\wedge\; v.\top \;\mapsto\; u.a \subset A \;\vee\; u.\bot$

, the first disjunct in the right side is *false* because $v \in u.a \;\equiv\; v \; R \; u$ and $v \; R \; u \;\Rightarrow\; \neg v.\bot$ (from Corollary 0)

$u.a = A \;\wedge\; \langle \exists\, v \;::\; v \in A \;\wedge\; v.\top \rangle \;\mapsto\; u.a \subset A \;\vee\; u.\bot$

, disjunction of the above over all v

$u.a = A \;\wedge\; A \neq \emptyset \;\mapsto\; u.a \subset A \;\vee\; u.\bot$ (2)

, a property of acyclic graphs is,
$u.a = A \;\wedge\; A \neq \emptyset \;\Rightarrow\; \langle \exists\, v \;::\; v \in A \;\wedge\; v.\top \rangle$

$u.a = A \;\wedge\; A = \emptyset \;\Rightarrow\; u.\top$

, definition of $u.a$ and $u.\top$

$u.a = A \;\wedge\; A = \emptyset \;\mapsto\; u.\bot$

, from the above and $u.\top \;\mapsto\; u.\bot$ (property P2)

$u.a = A \;\mapsto\; u.a \subset A \;\vee\; u.\bot$

, disjunction of the above and (2)

$(\mathit{true} \wedge\; u.a = A) \;\mapsto\; (\mathit{true} \wedge\; u.a \subset A) \;\vee\; u.\bot$

, rewriting the above

$\mathit{true} \;\mapsto\; u.\bot$

, induction rule on the above; a family of finite sets if well-founded under the subset ordering ▽

6 References

1. K. Mani Chandy and Jayadev Misra, *Parallel Program Design: A Foundation*, Addison-Wesley, 1988.