A Composition Theorem About Fixed Points

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Let F, G be arbitrary programs, T, p, q be predicates, and $F.FP$ be the fixed point predicate of F.

We prove three results of the following from where $\circ$ is *unless*, *ensures*, or $\mapsto$.

$$\frac{\begin{array}{l} T \;\Rightarrow\; F.FP, \\ T \text{ is stable in } G, \\ p \;\circ\; q \text{ in } G \end{array}}{(T \;\wedge\; p) \;\circ\; (T \;\wedge\; q) \text{ in } F \;[\!]\; G}$$

Theorem 1:

$$\frac{\begin{array}{l} T \;\Rightarrow\; F.FP, \\ T \text{ is stable in } G, \\ p \textit{ unless } q \text{ in } G \end{array}}{T \;\wedge\; p \;\; \textit{unless} \;\; T \;\wedge\; q \text{ in } F \;[\!]\; G}$$

Proof:

$(T \wedge p) \;\wedge\; F.FP$ is stable in F	, stability at fixed point (Section 3.6.4 of [1])	
$T \;\wedge\; p$ is stable in F	, $T \;\wedge\; F.FP = T$ from $T \;\Rightarrow\; F.FP$	(1)
p *unless* q in G	, given	
T is stable in G	, given	
$T \;\wedge\; P$ *unless* $T \;\wedge\; q$ in G	, conjunction of the above two	
$T \;\wedge\; p$ *unless* $T \;\wedge\; q$ in $F \;[\!]\; G$	, corollary to the union theorem on the above and (1)	∇

Theorem 2:

$$\frac{\begin{array}{l} T \;\Rightarrow\; F.FP, \\ T \text{ is stable in } G, \\ p \textit{ ensures } q \text{ in } G \end{array}}{T \;\wedge\; p \textit{ ensures } T \;\wedge\; q \text{ in } F \;[\!]\; G}$$

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Proof: Similar to the proof of Theorem 1; replace *unless* by *ensures*. ▽

Theorem 3:

$$\frac{\begin{array}{l} T \;\Rightarrow\; F.FP, \\ T \text{ is stable in } G, \\ p \;\mapsto\; q \text{ in } G \end{array}}{T \,\wedge\, p \;\mapsto\; T \,\wedge\, q \text{ in } F \,[\!]\, G}$$

Proof: We consider the three possible ways in which $p \mapsto q$ in G could have been proven, and we apply induction on the number of *leads-to* inference rules used in this proof.

Case 1) p *ensures* q in G:
- $T \wedge p$ *ensures* $T \wedge q$ in $F \,[\!]\, G$, using Theorem 2
- $T \wedge p \mapsto T \wedge q$ in $F \,[\!]\, G$, using definition of $\mapsto$

Case 2) $p \mapsto r$ in G, $r \mapsto q$ in G:
- $T \wedge p \mapsto T \wedge r$ in $F \,[\!]\, G$, induction hypothesis
- $T \wedge r \mapsto T \wedge q$ in $F \,[\!]\, G$, induction hypothesis
- $T \wedge p \mapsto T \wedge q$ in $F \,[\!]\, G$, transitivity on the above two

Case 3) $\langle \forall\, m \,::\, p.m \mapsto q \text{ in } G \rangle$ and $p = \langle \exists\, m \,::\, p.m \rangle$
- $\langle \forall\, m \,::\, T \wedge p.m \mapsto T \wedge q \text{ in } G \rangle$, induction hypothesis
- $\langle \exists\, m \,::\, T \wedge p.m \rangle \mapsto T \wedge q$ in G , disjunction on the above
- $T \wedge \langle \exists\, m \,::\, p.m \rangle \mapsto T \wedge q$ in G , m is not free in T
- $T \wedge p \mapsto T \wedge q$ in G , $p = \langle \exists\, m \,::\, p.m \rangle$ ▽

References

1. K. Mani Chandy and Jayadev Misra, *Parallel Program Design: A Foundation*, Addison-Wesley, 1988.