Functions Preserved by Unless/Leads-to

Notes on UNITY: 08-89

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6/16/89

In many proofs a standard paradigm is used to deduce from

$$x = m \textit{ unless } x > m$$

that

$$f(x) = n \textit{ unless } f(x) > n$$

holds, for a specific function f. For instance, given that

$$(x, y) = (a, b) \textit{ unless } (x, y) > (a, b)$$

where x, y are integers and $(x, y) > (a, b)$ stands for

$$(x \geq z \ \wedge \ y \geq b) \ \wedge \ (x > a \ \vee \ y > b)$$

we can deduce

$$x + y = m \textit{ unless } x + y > m$$

The purpose of this note is to set down conditions under which such a deduction can be made, for *unless* and *leads-to* properties.

Let A, B be sets in each of which a binary relation $\sim$ is defined (we use the same symbol for both relations). Let $f : A \rightarrow B$ be a function that preserves $\sim$, i.e., for any a, b from A,

$$a \sim b \Rightarrow f(a) \sim f(b).$$

Theorem 1: In the following, m, n are not free in p, q.

$$\frac{\langle \forall\ m \ : \ m \in A \ :: \ p \ \wedge \ x = m \textit{ unless } q \ \wedge \ x \ \sim \ m \rangle}{\langle \forall\ n \ : \ n \in B \ :: \ p \ \wedge \ f(x) = n \textit{ unless } q \ \wedge \ f(x) \ \sim \ n \rangle}$$

*This work was partially supported by ONR Contracts N00014-87-K-0510 and N00014-86-0763 and by a grant from The John Simon Guggenheim Foundation.

Proof: Choose an arbitrary n from B. Now we have from the antecedent,

$$\langle \forall\ m\ :\ m \in A\ ::\ p\ \wedge\ x = m\ \textit{unless}\ q\ \wedge\ x\ \sim\ m \rangle$$

We apply simple disjunction over all m where $f(m) = n$.

$$\langle \exists\ m\ :\ f(m) = n\ ::\ p\ \wedge\ x = m \rangle\ \textit{unless}\ \langle \exists\ m\ :\ f(m) = n\ ::\ q\ \wedge\ x\ \sim\ m \rangle$$

lhs
$\equiv \langle \exists\ m\ ::\ f(m) = n\ \wedge\ p\ \wedge\ x = m \rangle$
$\equiv p\ \wedge\ f(x) = n$, m is not free in p
rhs
$\equiv \langle \exists\ m\ :\ f(m) = n\ ::\ q\ \wedge\ x\ \sim\ m \rangle$
$\equiv q\ \wedge\ \langle \exists\ m\ :\ f(m) = n\ ::\ x\ \sim\ m \rangle$, m is not free in q
$\Rightarrow q\ \wedge\ \langle \exists\ m\ :\ f(m) = n\ ::\ f(x)\ \sim\ f(m) \rangle$, $x\ \sim\ m\ \Rightarrow\ f(x)\ \sim\ f(m)$
$\Rightarrow q\ \wedge\ \langle \exists\ m\ ::\ f(x)\ \sim\ n \rangle$, Leibniz
$\equiv q\ \wedge\ f(x)\ \sim\ n$ □

A similar result for *leads-to* is in the following.

Theorem 2: In the following, m, n are not free in p, q.

$$\frac{\langle \forall\ m\ :\ m \in A\ ::\ p\ \wedge\ x = m\ \mapsto\ q\ \wedge\ x\ \sim\ m \rangle}{\langle \forall\ n\ :\ n \in B\ ::\ p\ \wedge\ f(x) = n\ \mapsto\ q\ \wedge\ f(x)\ \sim\ n \rangle}$$

Proof: Apply general disjunction rule to the antecedent, for all m such that $f(m) = n$, to conclude

$$\langle \exists\ m\ :\ f(m) = n\ ::\ p\ \wedge\ x = m \rangle\ \mapsto\ \langle \exists\ m\ :\ f(m) = n\ ::\ q\ \wedge\ x\ \sim\ m \rangle$$

lhs $\equiv$ $p\ \wedge\ f(x) = n$ {see the last proof}
rhs $\Rightarrow$ $q\ \wedge\ f(x)\ \sim\ n$ {see the last proof} □