Monotonicity, Stability and Constants

Notes on UNITY: 10-89

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An underexploited concept of UNITY is *constant*. A predicate was defined to be constant—see Section 3.4.3 of the book—if the predicate and its negation are both stable. This definition can be extended to arbitrary expressions: expression e is constant if for all possible values m, $e = m$ is stable. (It follows that a constant predicate is stable.) Some of the useful properties of constants are given below.

(Constant Definition)

If x is not modified in program F then x is constant in F.

Thus, all local variables of program G are constant in F whenever F, G are disjoint (i.e., F, G do not share statements).

(Constant Formation)

Any expression of constants and free variables is constant.

Note that a free variable merely indicates that the property can be instantiated with all possible values of the free variable; all such instantiations yield expressions consisting of constants only and hence the expression is constant.

Notation: Henceforth, m, n, k are free variables and x, u, v are program variables.

(Constant Introduction) For a function f over x,

$$\frac{x = m \;\; \textit{unless} \;\; x \neq m \;\wedge\; f(x) = f(m)}{f(x) \text{ constant}}$$

The constant introduction rule is quite powerful. To see an application, suppose that for integer valued program variables u, v

$$(u, v) = (m, n) \;\; \textit{unless} \;\; (u, v) = (m-1, n+1) \;\vee\; (u, v) = (m+1, n-1)$$

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Then we may conclude that $u+v$ is constant because we have,

$$(u,v)=(m,n) \ \ \textit{unless} \ \ (u,v)\neq(m,n) \ \wedge \ u+v=m+n$$

We prove the constant introduction rule in this note as a special case of a more general result on *monotonicity*, treated next.

Monotonicity An expression e is *monotone* with respect to a transitive relation $\sim$ means, for all possible values m of e,

$$e \ \sim \ m \ \ \text{stable}$$

That is, e never "decreases" in the relation $\sim$. Hence

$$e \text{ constant} \quad \equiv \quad e \text{ is monotone with respect to } =$$

(Note that "=" is transitive.)

Theorem (Monotonicity)

For a transitive relation $\sim$

$$\frac{x=m \ \ \textit{unless} \ \ x\neq m \ \wedge \ f(x) \ \sim \ f(m)}{f(x) \text{ is monotone with respect to } \ \sim}$$

Proof: See appendix. □

Corollary 1 (Constant Introduction)

$$\frac{x=m \ \ \textit{unless} \ \ x\neq m \ \wedge \ f(x)=f(m)}{f(x) \text{ constant}}$$

Proof: Set "$\sim$" to "=" in the theorem and use the definition of constant. □

Corollary 2: For a predicate p over x

$$\frac{x=m \ \ \textit{unless} \ \ x\neq m \ \wedge \ p(x)}{p(x) \text{ stable}}$$

Proof: The term $p(x)$ is the rhs of the antecedent can be weakened to $p(x) \ \Leftarrow \ p(m)$ where "$\Leftarrow$," called "follows from," is transitive. Hence, from the theorem,

$$\begin{array}{lll} p(x) \ \Leftarrow \ k & \text{stable} & \\ p(x) \ \Leftarrow \ \textit{true} & \text{stable} & \text{, setting } k \text{ to } \textit{true} \\ p(x) & \text{stable} & \text{, } p(x) \ \equiv \ [p(x) \ \Leftarrow \ \textit{true}] \end{array}$$

□

Corollary 3: Let ">" be an irreflexive, transitive relation. Suppose a function f satisfies

$$m>n \ \Rightarrow \ f(m)>f(n)$$

{There are two different relations ">"—one in the domain of f and the other in the range of f. We use the same symbol for both.}

$$\frac{x=m \ \ \textit{unless} \ \ x>m}{f(x) \text{ is monotone with respect to } >}$$

Proof:

$x = m$ *unless* $x > m$, antecedent

$x = m$ *unless* $x \neq m \ \wedge \ x > m$, irreflexivity of “$>$”

$x = m$ *unless* $x \neq m \ \wedge \ f(x) > f(m)$, weaken rhs using $x > m \ \Rightarrow \ f(x) > f(m)$

f is monotone with respect to $>$, from the theorem using transitivity of “$>$” □

Corollary 4:

$$\frac{x = m \ \ \textit{unless} \ \ x > m}{x \text{ is monotone with respect to } >}$$

Proof: Set f to the identity function in Corollary 3. □

Appendix: Proof of the Main Theorem

We prove a more general result.

Theorem: In the following, m does not occur free in q. Let $\sim$ be transitive.

$$\frac{x = m \ \ \textit{unless} \ \ [x \neq m \ \wedge \ f(x) \ \sim \ f(m)] \ \vee \ q}{f(x) \ \sim \ k \ \ \textit{unless} \ \ q}$$

Proof:

Consider any arbitrary k. Take disjunction of the antecedent over all m where $f(m) \ \sim \ k$. Applying the disjunction rule—see Notes on UNITY 01–88—gives us

$$\begin{array}{l} \langle \exists \ m \ : \ f(m) \ \sim \ k \ :: \ x = m \rangle \\ \qquad \textit{unless} \\ \quad \langle \forall \ m \ : \ f(m) \ \sim \ k \ :: \ x \neq m \ \vee \ [x \neq m \ \wedge \ f(x) \ \sim \ f(m)] \ \vee \ q \rangle \\ \wedge \ \langle \exists \ m \ : \ f(m) \ \sim \ k \ :: \ [x \neq m \ \wedge \ f(x) \ \sim \ f(m)] \ \vee \ q \rangle \end{array}$$

The lhs $\equiv \ f(x) \ \sim \ k$

The first term in the rhs

$\equiv \ \langle \forall \ m \ : \ f(m) \ \sim \ k \ :: \ x \neq m \ \vee \ q \rangle$

$\equiv \ q \ \vee \ \langle \forall \ m \ : \ f(m) \ \sim \ k \ :: \ x \neq m \rangle$

$\equiv \ q \ \vee \ \neg [f(x) \ \sim \ k]$

The second term in the rhs

$\Rightarrow \ \langle \exists \ m \ :: \ f(m) \ \sim \ k \ \wedge \ f(x) \ \sim \ f(m) \rangle \ \vee \ q$

$\Rightarrow$ {using transitivity of $\sim$} $\langle \exists \ m \ :: \ f(x) \ \sim \ k \rangle \ \vee \ q$

$\equiv \ f(x) \ \sim \ k \ \vee \ q$

Hence rhs

$\Rightarrow \ (q \ \vee \ \neg [f(x) \ \sim \ k]) \ \wedge \ (f(x) \ \sim \ k \ \vee \ q)$

$\equiv \ q$

Combining the lhs and the rhs, we obtain

$f(x) \ \sim \ k$ *unless* q □

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