A Generalization of the Completion Theorem

Notes on UNITY: 29–91

Jayadev Misra*

Department of Computer Sciences

The University of Texas at Austin

Austin, Texas 78712

(512) 471-9547

misra@cs.utexas.edu

12/20/91

The completion theorem is the following. For any finite set of predicates, $p.i$s and $q.i$s,

$$\frac{(\forall\ i\ ::\ \ p.i \mapsto q.i \quad q.i \textit{ unless } b\ )}{(\forall\ i\ ::\ p.i) \mapsto (\forall\ i\ ::\ q.i) \lor b}$$

We prove the following generaliztion.

Theorem

$$\frac{(\forall\ i\ ::\ \ p.i \mapsto q.i \quad q.i \textit{ unless } b\ )}{(\forall\ i : i \in S : p.i) \mapsto [(\forall\ i : i \in S : q.i) \land (\forall\ i : i \notin S : p.i \Rightarrow q.i)] \lor b}$$

Here S is any subset of predicate indices.

Corollary 1: Setting S to the complete set (of all indices) we obtain the completion theorem.

Corollary 2: Setting S to the empty set we obtain as the consequent,

$$\textit{true} \mapsto (\forall\ i\ ::\ p.i \Rightarrow q.i) \lor b$$

Corollary 3: From the given antecedents, conclude

$$(\exists\ i\ ::\ p.i) \mapsto [(\forall\ i\ ::\ p.i \Rightarrow q.i) \land (\exists\ i\ ::\ q.i)] \lor b$$

*This material is based in part upon work supported by the Texas Advanced Research Program under Grant No. 003658–065, by the Office of Naval Research Contract N00014-90-J-1640 and by the National Science Foundation Award CCR–9111912.

Proof: From the theorem

$$p.j \mapsto [q.j \;\wedge\; (\forall\, i \;:\; i \neq j \;:\; p.i \;\Rightarrow\; q.i)] \;\vee\; b$$
$$\Rightarrow [(\forall\, i \;\;::\;\; p.i \;\Rightarrow\; q.i) \;\wedge\; (\exists\, i \;\;::\;\; q.i)] \;\vee\; b$$

The result follows by taking the disjunction of the above *leads-to* over all j.

Note: The first disjunct in the rhs is the same as the rhs of "*unless* -disjunction"; see UNITY–01.

Proof of the Theorem (Due to Ernie Cohen)

Lemma: From the given antecendents

$$\mathit{true} \;\mapsto\; (\forall\, i \;\;::\;\; p.i \;\Rightarrow\; q.i) \;\vee\; b$$

Proof: Assume that the predicate indices range over $0 \leq i < N$. Define

$$A.j \;\equiv\; (\forall\, i \;:\; 0 \leq i < j \;:\; p.i \;\Rightarrow\; q.i)$$

We show, $(\forall\, j \;:\; 0 \leq j \leq N \;:\; \mathit{true} \;\mapsto\; A.j \;\vee\; b)$. It then follows that $\mathit{true} \;\mapsto\; A.N \;\vee\; b$, which is the statement of the lemma. The proof is by induction on j.

$j = 0 \;:\; \mathit{true} \;\mapsto\; \mathit{true}$, follows trivially
$j+1, j \geq 0 \;:$, *true*
$\mapsto$ {induction hypothesis} $A.j \;\vee\; b$
$\Rightarrow$ {weakening} $(A.j \;\wedge\; \neg p.j) \;\vee\; p.j \;\vee\; b$
$\mapsto$ {$p.j \;\mapsto\; q.j$, cancellation} $(A.j \;\wedge\; \neg p.j) \;\vee\; q.j \;\vee\; b$
$\mapsto$ {$q.j \;\mapsto\; (A.j \;\wedge\; q.j) \;\vee\; b$, applying PSP to $\mathit{true} \;\mapsto\; A.j \;\vee\; b$, $q.j$ *unless* b}
$(A.j \;\wedge\; \neg p.j) \;\vee\; (A.j \;\wedge\; q.j) \;\vee\; b$
$\equiv$ {rewriting} $[A.j \;\wedge\; (p.j \;\Rightarrow\; q.j)] \;\vee\; b$
$\equiv$ {definition of $A.(j+1)$} $A.(j+1) \;\vee\; b$ □

The proof of the main theorem is as follows.

$\mathit{true} \;\mapsto\; (\forall\, i \;::\; p.i \;\Rightarrow\; q.i) \;\vee\; b$
, from the lemma
$(\forall\, i \;:\; i \in S \;:\; q.i)$ *unless* b
, simple conjunction over the *unless* properties
$(\forall\, i \;:\; i \in S \;:\; q.i) \mapsto [(\forall\, i \;::\; p.i \;\Rightarrow\; q.i) \;\wedge\; (\forall\, i \;:\; i \in S \;:\; q.i)] \;\vee\; b$
, PSP on the above two
$(\forall\, i \;:\; i \in S \;:\; p.i) \mapsto (\forall\, i \;:\; i \in S \;:\; q.i)$
, completion theorem
$(\forall\, i \;:\; i \in S \;:\; p.i) \mapsto [(\forall\, i \;::\; p.i \;\Rightarrow\; q.i) \;\wedge\; (\forall\, i \;:\; i \in S \;:\; q.i)] \;\vee\; b$
, transitivity on the above two
$(\forall\, i \;:\; i \in S \;:\; p.i) \mapsto [(\forall\, i \;:\; i \in S \;:\; q.i) \;\wedge\; (\forall\, i \;:\; i \notin S \;:\; p.i \;\Rightarrow\; q.i)] \;\vee\; b$
, predicate calculus on the rhs □