More on *detects* and *trails*

Notes on UNITY: 30–91

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In a given program, for predicates p, q define p *tracks* q to mean

- whenever p holds, so does q, i.e.,

  $p \;\Rightarrow\; q$ invariant

- once p holds it continues to hold as long as q holds, i.e.,

  p *unless* $\neg q$

- if q remains *true* forever then p holds eventually, i.e.,

  $q \;\mapsto\; \neg q \;\vee\; p$

Note: The last property is equivalent to

$$\mathit{true} \;\mapsto\; \neg q \;\vee\; p$$

Using the fact that $p \;\Rightarrow\; q$, we may rewrite this as

$$\mathit{true} \;\mapsto\; p \;\equiv\; q$$

Note: The relation *detects*—introduced in Chapter 9 of [**?**]—omitted p *unless* $\neg q$; therefore, Theorem 2, given below, does not hold for *detects*. The relations *trails*—introduced in UNITY–18—had the first two properties, but its progress condition was

$$q \;\mapsto\; p$$

Again, Theorem 2 does not hold for *trails*. The weakening of the progress condition allows us to apply *tracks* to a larger class of problems (*viz.*, termination detection in systems where idleness of any individual process is not stable). □

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Note: Given that q is stable and p *tracks* q, we can deduce:

$p \;\Rightarrow\; q$, p stable, $q \;\mapsto\; p$ □

Theorem 1: *tracks* is reflexive, antisymmetric and transitive.

Proof:

tracks is reflexive: show p *tracks* p
- $p \;\Rightarrow\; p$, predicate calculus
- p *unless* $\neg p$, antireflexivity of *unless*
- $p \;\mapsto \neg p \;\vee\; p$, implication

tracks is antisymmetric: given p *tracks* q and q *tracks* p:
- $p \;\Rightarrow\; q$, from p *tracks* q
- $q \;\Rightarrow\; p$, from q *tracks* p
- $p \;\equiv\; q$, from the above two

tracks is transitive: given p *tracks* q and q *tracks* r:
- • $p \;\Rightarrow\; q$, from p *tracks* q
- $q \;\Rightarrow\; r$, from q *tracks* r
- $p \;\Rightarrow\; r$, from the above two
- • p *unless* $\neg q$, from p *tracks* q
- q *unless* $\neg r$, from q *tracks* r
- $p \;\wedge\; q$ *unless* $\neg r$, conjunction and weakening the rhs
- p *unless* $\neg r$, substitution axiom on lhs with $p \;\Rightarrow\; q$ invariant
- • $q \;\mapsto\; \neg q \;\vee\; p$, from p *tracks* q
- q *unless* $\neg r$, from q *tracks* r
- $q \;\mapsto\; (p \;\wedge\; q) \;\vee\; \neg r$, PSP
- $q \;\mapsto\; p \;\vee\; \neg r$, weakening the rhs
- $r \;\mapsto\; \neg r \;\vee\; q$, from q *tracks* r
- $r \;\mapsto\; \neg r \;\vee\; p$, cancellation on the above two □

The following theorem allows tracking a predicate by tracking each of its conjuncts individually.

Theorem 2:

$$\frac{p \text{ tracks } q \;,\; p' \text{ tracks } q'}{p \;\wedge\; p' \text{ tracks } q \;\wedge\; q'}$$

Proof:

1. To show $p \;\wedge\; p' \;\Rightarrow\; q \;\wedge\; q'$
   - $p \;\Rightarrow\; q$, from p *tracks* q
   - $p' \;\Rightarrow\; q'$, from p' *tracks* q'
   - $p \;\wedge\; p' \;\Rightarrow\; q \;\wedge\; q'$, from the above two
2. To show $p \;\wedge\; p'$ *unless* $\neg(q \;\wedge\; q')$
   - p *unless* $\neg q$, from p *tracks* q
   - p' *unless* $\neg q'$, from p' *tracks* q'
   - $p \;\wedge\; p'$ *unless* $\neg q \;\vee\; \neg q'$, simple conjunction
3. To show $q \;\wedge\; q' \;\mapsto\; \neg(q \;\wedge\; q') \;\vee\; (p \;\wedge\; p')$
   - p *unless* $\neg q$, from p *tracks* q
   - $\neg q \;\vee\; p$ *unless* $\neg q$, simple disjunction with $\neg q$ *unless* $\neg q$
   - $\neg q \;\vee\; p$ *unless* $\neg q \;\vee\; \neg q'$, weakening the rhs

$\neg q' \lor p'$ *unless* $\neg q \lor \neg q'$	, similarly
$q \mapsto \neg q \lor p$	, from p *tracks* q
$q' \mapsto \neg q' \lor p'$	, from p' *tracks* q'
$q \land q' \mapsto [(\neg q \lor p) \land (\neg q' \lor p')] \lor \neg q \lor \neg q'$	, completion on the above four
$q \land q' \mapsto \neg(q \land q') \lor (p \land p')$	, simplifying the rhs $\square$

The following theorem shows that a tracking predicate inherits the properties of the tracked predicate.

Theorem 3

$$\frac{p \textit{ tracks } q \ , \ q \circ r}{p \circ r} \qquad \text{where } \circ \text{ is } \textit{unless}, \textit{ensures} \text{ or } \mapsto$$

Proof:

unless :	p *unless* $\neg q$	, from p *tracks* q
	q *unless* r	, given
	$p \land q$ *unless* r	, conjunction and weakening the rhs
	p *unless* r	, $p \Rightarrow q$ from p *tracks* q
ensures	: similar to the above proof	
$\mapsto$:	$p \Rightarrow q$	, from p *tracks* q
	$p \mapsto r$	, given
	$p \mapsto r$	, implication and transitivity $\square$