Another Theorem on Strengthening the Guard
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It is well known that all safety properties of a program are preserved if the guard of any of its statements is strengthened. This note develops a result about progress properties. Also, see Notes on UNITY, 19, “More on Strengthening the Guard”.

Let $F$ be a program which includes a statement $t$ with guard $q$. Let $G$ be a program obtained by strengthening the guard of $t$.

**Theorem** \[
\frac{p \implies r \text{ in } F}{p \implies q \lor r \text{ in } G}
\]

Proof: By structural induction on the proof of $p \implies r$ in $F$.

- $p \text{ en } r$ in $F$: Then, $p \land \neg r \text{ co } p \lor r$ in $F$, which also holds in $G$, because all safety properties are preserved by strengthening the guard. Next, from the definition of $\text{ en }$, there exists some action $s$ in $F$ so that
  \[
  \{p \land \neg r\} s \{r\}
  \]
  If $s \neq t$ then this assertion holds in $G$. If $s = t$ then
  \[
  \{p \land \neg r\} t \{r\}
  \]

  Thus, execution of $t$ in $F$ in any state satisfying $p \land \neg r$ causes a state change, i.e., $t$ executes effectively. Since the guard of $t$ is $q$,

  \[
p \land \neg r \implies q, \text{ or} \\
p \implies q \lor r
  \]

  Therefore, $p \implies q \lor r$ in $G$.

- $p \equiv p' \lor p''$, where $p' \implies r$ and $p'' \implies r$ in $F$: Using induction,

  \[
p' \implies q \lor r \text{ in } G \\
p'' \implies q \lor r \text{ in } G. \text{ Using disjunction,} \\
p \implies q \lor r \text{ in } G
  \]

- $p \implies p' \implies r$ in $F$: Using induction,

  \[
p \implies q \lor p' \text{ in } G \\
p' \implies q \lor r \text{ in } G. \text{ Using cancellation,} \\
p \implies q \lor r \text{ in } G
  \]

**Corollary** Let $q$ be the guard of a statement in $F$ which is strengthened to obtain program $G$. Then,

\[
p' \implies q \text{ in } F \\
p \implies q \text{ in } G
\]