A Constructive Proof of Vizing's Theorem

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We consider finite graphs with no self-loops and no multiple edges. A graph is *valid* if all edges incident on a vertex have different colors. We prove

Vizing's Theorem. All the edges of a graph of maximum degree less than N can be colored using N colors so that the graph is valid.

We call a color *incident on* a vertex if an edge incident on that vertex has that color; otherwise, the color is *free* on that vertex. Since there are N available colors and each vertex degree is less than N, there is a free color on each vertex.

Our proof consists of showing how to color an arbitrary uncolored edge of a valid graph (which may require changing the colors of already-colored edges to maintain validity). This procedure can be repeated until all edges are colored. Henceforth, $X\,Y$ is the uncolored edge that is to be colored.

The fan

Our proof uses a data structure called a *fan*. A fan $\langle f..l \rangle$ is a sequence of vertices that satisfy the following (throughout the paper, u ranges over all elements of the fan except l, and u^+ is the successor of u in the fan):

$F0 : \langle f..l \rangle$ is a nonempty sequence of distinct neighbors of X;

$F1 :$ edge $X\,f$ is uncolored; and

$F2 : (\forall u ::$ color of edge $X\,u^+$ is free on $u)$.

Each edge $X\,v$ for $v \in \langle f..l \rangle$ is called a *fan edge*.

The following (nondeterministic) algorithm constructs a fan F that is maximal, in that it cannot be extended:

$$F := \langle Y \rangle;\ \textbf{do}\ (\exists v :: B) \rightarrow F := F\ \textbf{cat}\ v\ \textbf{where}\ v\ \textbf{satisfies}\ B\ \textbf{od}$$

where B is the predicate

$B : X\,v$ is an edge $\wedge$ $v \notin F$ $\wedge$

[1]This work was done while the second author was on sabbatic from Cornell at Austin.

color of $X\,v$ is free on the last vertex of F .

The algorithm

Our algorithm for coloring edge $X\,Y$ is given below. Subsequently, we investigate the steps of the algorithm.

Let $\langle f..l \rangle$ be a maximal fan;
Let c be a color that is free on X and d a color that is free on l;
Invert the *cd*-path;
$\{d$ free on $X \;\;\wedge\;\;$ the graph is valid $\;\;\wedge$
$\quad(\exists w :: w \in \langle f..l \rangle \;\;\wedge\;\; \langle f..w \rangle$ is a fan $\;\wedge\;\; d$ free on $w)\}$
Let w satisfy: $w \in \langle f..l \rangle$, $\langle f..w \rangle$ is a fan, and d free on w;
Rotate fan $\langle f..w \rangle$ and give edge $X\,w$ the color d

Inverting the cd-path

A *cd*-path is a path that includes vertex X , has edges colored only c or d , and is maximal —it cannot be extended with a *c*- or *d*-colored edge. Since the graph is finite and valid, since c is free on X , and since the *cd*-path is maximal, we conclude that colors c and d alternate along successive edges of a *cd*-path, that the *cd*-path is simple and unique, and that X is an endpoint of it. Further, if the *cd*-path contains an edge, it has two endpoints and the edge of it that is incident on X has color d .

Operation *Invert the cd-path* is:

Switch the colors of the edges on the *cd*-path: c to d and d to c.

Inversion is performed to make d free on X , so that some edge incident on X can be given the color d . Since c is free on X prior to the inversion, either d is free on X initially and remains free or the color of an edge incident on X is changed from c to d , thus freeing d on X .

Inversion maintains the validity of the graph because: the bag of incident colors on each vertex outside the *cd*-path is unchanged; the bag of incident colors on each internal vertex of the *cd*-path is unchanged (since each has incident on it one *c*-edge and one *d*-edge); and the new color (either c or d) of the edge of the *cd*-path incident on each endpoint was previously free on that endpoint.

We show that inversion guarantees the existence of a vertex w in $\langle f..l\rangle$ such that $\langle f..w\rangle$ is a fan and d is free on w. We consider two cases, prior to the inversion.

Case 0: No fan edge has color d. Since fan $\langle f..l\rangle$ is maximal and d is free on l, no edge with color d is incident on X. Hence, d is free on X. Hence, the *cd*-path is empty and the inversion has no effect on the graph or fan. Choose $w = l$.

Case 1: A fan edge has color d. Since $X\,f$ is uncolored, this fan edge differs from $X\,f$; call it $X\,v^+$. By $F2$, d is free on vertex v. Note that v differs from l because l is the last vertex in the fan and v^+ is in the fan. We now show that after the inversion

$$F3 : (\forall u : u \neq v : \text{color of } X\,u^+ \text{ free on } u) \quad .$$

holds. Prior to the inversion, the color of $X\,u^+$ for $u \neq v$ is neither c (since c is free on X) nor d (since $X\,v^+$ has color d and the graph is valid). Inversion changes only edges colored c and d. Therefore, the color of $X\,u^+$ and the freeness of the color of $X\,u^+$ on u remains unchanged, so $F3$ holds after the inversion.

Either v is in the *cd*-path or not. If not, the inversion does not affect the bag of colors free on v, and d remains free on v. Since a prefix of a fan is a fan, $\langle f..v\rangle$ is a fan, and we can choose $w = v$.

Suppose v is in the *cd*-path. We show that $\langle f..l\rangle$ remains a fan and that d remains free on l, so we can choose $w = l$. Since d is free on v before the inversion, v is an endpoint of a *cd*-path. Inversion changes the color of $X\,v^+$ from d to c and makes c free on v; together with $F3$, this implies that $F2$ holds after the inversion. Thus, $\langle f..l\rangle$ remains a fan. Color d remains free on l because l is not on the *cd*-path (since d is free on l, l would have to be an endpoint of the *cd*-path, but its endpoints are X and v, which are different from l).

Rotating fan $\langle f..w\rangle$

Operation *Rotate fan* $\langle f..w\rangle$ is as follows (in it, u ranges over all the elements of $\langle f..w\rangle$ except w):

In parallel, give each edge $X\,u$ the color of $X\,u^+$ and uncolor $X\,w$.

This operation leaves the graph valid because: the bag of colors incident on a vertex outside $\langle f..w\rangle$ (including X) is unchanged; the bag of colors incident on w is reduced (since $X\,w$ is uncolored); and, by $F2$, each edge Xu has a new incident color that was free on u prior to the rotation.

Rotation leaves d free on w, since the only edge incident on w that has its color changed by the rotation is $X\,w$ and this edge is uncolored by the rotation. Since d is also free on X, and since the rotation uncolors $X\,w$, giving $X\,w$ the color d increases the number of colored edges and leaves the graph valid.

1 Discussion

The study of Vizing's theorem was begun by E.W. Dijkstra and the ATAC (Austin Tuesday Afternoon Club) at the request of Bob Tarjan, who had found the existing proofs of the theorem too confusing and complex. A constructive proof, as explained by J.R. Rao to the ATAC in February 1990, contained the essential ingredients —the fan, inversion, and rotation— but without the nomenclature. Further, it was more complicated. Rao and Dijkstra then derived a quite different algorithm, which has the nice property that its derivation can be explained [2]. The simple view of the algorithm presented here, which at its top level contains no branches, is due to many attempts at simplification, mostly by Misra.

The algorithm in in [1, p. 94], which we looked at after completing our proof, is similar to ours but more complicated. In his algorithm the inversion and rotation occur in a different order. Bollobás's proof is only one page long, the shortness being achieved through omission of necessary arguments. For example, Bollobás neglects even to mention that inversion and rotation leave the graph valid, and those arguments are not obvious. On the other hand, we have attempted to include all necessary arguments and to be as clear as possible. Bollobás's proof is complicated somewhat by the notation and by the use of subgraphs formed by edges of two colors instead of simply the *cd*-paths.

References

[1] Bollobás, B. *Graph Theory.* Springer Verlag, New York, 1979.

[2] Dijkstra, E.W., and J.R. Rao. Designing the proof of Vizing's algorithm. EWD1082a, Computer Sciences, University of Texas at Austin, September 1990.