Knuth's Generalization of Takeuchi's Tarai Function

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Takeuchi's Tarai Function

For integer inputs, x, y, z,

$$t(x,y,z) \overset{\text{def}}{\Leftarrow} \text{ if } \ x \leq y \ \text{ then } \ y \\ \text{ else } \ t(\ t(x-1,y,z), \\ t(y-1,z,x), \\ t(z-1,x,y)\ ).$$

John McCarthy proved this recursion terminates and t can be computed without any recursion,

$$t(x,y,z) \ = \ \text{if } \ x \leq y \ \text{ then } \ y \\ \text{else if } \ y \leq z \ \text{ then } \ z \\ \text{else } \ x.$$

McCarthy's Measure

$$\text{measure}(x, y, z) \stackrel{\text{def}}{\Leftarrow} \text{measure}_1(x - y, z - y)$$

where

$$\begin{aligned}
&\text{measure}_1(m, n) \stackrel{\text{def}}{\Leftarrow} \\
&\quad \text{if } \; m \leq 0 \; \text{ then } \; 0 \\
&\qquad \text{else if } \; n \geq 2 \\
&\qquad\qquad \text{then } \; m + n(n-1)/2 - 1 \\
&\qquad \text{else if } \; n \geq 0 \; \text{ then } \; m \\
&\qquad \text{else if } \; n = -1 \\
&\qquad\qquad \text{then } \; (m+1)(m+2)/2 - 1 \\
&\qquad \text{else } \; (m-n)(m-n+1)/2 - m - 1.
\end{aligned}$$

J Moore's Simpler Measure

Early Boyer-Moore theorem prover, THM, used to verify termination and to show t satisfies the simpler nonrecursive equation.

Lexicographical ordering on triples of nonnegative integers:

$$m(x,y,z) \stackrel{\text{def}}{\Leftarrow} \\ < m_1(x,y,z), m_2(x,y,z), m_3(x,y,z) >$$

where

$$\begin{aligned} m_1(x,y,z) &\stackrel{\text{def}}{\Leftarrow} \text{ if } x \le y \text{ then } 0 \text{ else } 1, \\ m_2(x,y,z) &\stackrel{\text{def}}{\Leftarrow} \max(x,y,z) - \min(x,y,z), \\ m_3(x,y,z) &\stackrel{\text{def}}{\Leftarrow} x - \min(x,y,z). \end{aligned}$$

Knuth's Generalization

Generalize the tarai function to higher dimensions:

For integer inputs, $x_1, x_2, \ldots, x_m$,

$$
\begin{aligned}
& t(x_1, x_2, \ldots, x_m) \stackrel{\text{def}}{\Leftarrow} \\
& \quad \text{if } \; x_1 \le x_2 \; \text{ then } \; x_2 \\
& \qquad \text{else } \; t(\; t(x_1 - 1, x_2, \ldots, x_m), \\
& \qquad\qquad\quad t(x_2 - 1, x_3, \ldots, x_m, x_1), \\
& \qquad\qquad\qquad \vdots \\
& \qquad\qquad\quad t(x_m - 1, x_1, \ldots, x_{m-1}) \;).
\end{aligned}
$$

Knuth's Two Questions

1. Are there total functions on the integers that satisfy the recursive equation based on the definition?

   That is, are there total functions $f(x_1, x_2, \ldots, x_m)$ on the integers that satisfy the equation

$$
\begin{aligned}
&f(x_1, x_2, \ldots, x_m) = \\
&\quad \text{if} \;\; x_1 \le x_2 \;\; \text{then} \;\; x_2 \\
&\quad\quad \text{else} \;\; f(\; f(x_1 - 1, x_2, \ldots, x_m), \\
&\quad\quad\quad\quad f(x_2 - 1, x_3, \ldots, x_m, x_1), \\
&\quad\quad\quad\quad \vdots \\
&\quad\quad\quad\quad f(x_m - 1, x_1, \ldots, x_{m-1}) \;)?
\end{aligned}
$$

2. Does the recursion terminate for all integer inputs?

Question 1: Can the recursive equation be satisfied?

McCarthy proved when $m = 3$:

The function

$$t(x_1, x_2, x_3) \stackrel{\text{def}}{\Leftarrow} \text{ if } x_1 \leq x_2 \text{ then } x_2 \text{ else if } x_2 \leq x_3 \text{ then } x_3 \text{ else } x_1$$

satisfies the equation

$$t(x_1, x_2, x_3) = \text{ if } x_1 \leq x_2 \text{ then } x_2 \text{ else } t(\, t(x_1 - 1, x_2, x_3), t(x_2 - 1, x_3, x_1), t(x_3 - 1, x_1, x_2)\,).$$

Question 1: Can the recursive equation be satisfied?

Knuth notes when $m = 4$:

The function

$$
\begin{aligned}
&t(x_1, x_2, x_3, x_4) \stackrel{\text{def}}{\Leftarrow} \\
&\quad \text{if } \ x_1 \le x_2 \ \text{ then } \ x_2 \\
&\qquad \text{else if } \ x_2 \le x_3 \ \text{ then } \ x_3 \\
&\qquad \text{else if } \ x_3 \le x_4 \ \text{ then } \ x_4 \\
&\qquad\qquad \text{else } \ x_1
\end{aligned}
$$

satisfies the equation

$$
\begin{aligned}
&t(x_1, x_2, x_3, x_4) = \\
&\quad \text{if } \ x_1 \le x_2 \ \text{ then } \ x_2 \\
&\qquad \text{else } \quad t(\ t(x_1 - 1, x_2, x_3, x_4), \\
&\qquad\qquad\quad t(x_2 - 1, x_3, x_4, x_1), \\
&\qquad\qquad\quad t(x_3 - 1, x_4, x_1, x_2), \\
&\qquad\qquad\quad t(x_4 - 1, x_1, x_2, x_3)\ ).
\end{aligned}
$$

Question 1: Can the recursive equation be satisfied?

Conjecture

The function

$$
\begin{aligned}
& t(x_1, x_2, \ldots, x_m) \stackrel{\text{def}}{\Leftarrow} \\
& \quad \text{if} \quad (\exists k < m)(x_1 > x_2 > \cdots > x_k \le x_{k+1}) \\
& \qquad \text{then} \quad x_{k+1} \\
& \qquad \text{else} \quad x_1
\end{aligned}
$$

satisfies the equation

$$
\begin{aligned}
& t(x_1, x_2, \ldots, x_m) = \\
& \quad \text{if} \quad x_1 \le x_2 \quad \text{then} \quad x_2 \\
& \qquad \text{else} \quad t(\; t(x_1 - 1, x_2, \ldots, x_m), \\
& \qquad\qquad\quad t(x_2 - 1, x_3, \ldots, x_m, x_1), \\
& \qquad\qquad\qquad \vdots \\
& \qquad\qquad\quad t(x_m - 1, x_1, \ldots, x_{m-1})\;).
\end{aligned}
$$

Question 1: Can the recursive equation be satisfied?

Knuth's Counterexample, when $m = 5$

Left side

$$t(5,3,2,0,1) = 1$$

Right side

$$\begin{aligned} t(5,3,2,0,1) &= t(\ t(4,3,2,0,1), \\ &\qquad t(2,2,0,1,5), \\ &\qquad t(1,0,1,5,3), \\ &\qquad t(-1,1,5,3,2), \\ &\qquad t(0,5,3,2,0)\ ) \\ &= t(1,2,1,1,5) \\ &= 2 \end{aligned}$$

Question 1: Can the recursive equation be satisfied?

Knuth proposes a solution

$$
\begin{aligned}
& f(x_1, x_2, \ldots, x_m) \stackrel{\text{def}}{\Leftarrow} \\
& \quad \text{if} \quad (\exists k < m)(x_1 > x_2 > \cdots > x_k \leq x_{k+1}) \\
& \quad\quad \text{then} \quad g(x_1, x_2, \ldots, x_{k+1}) \\
& \quad\quad \text{else} \quad x_1
\end{aligned}
$$

Here the "function" g takes a variable number (at least two) of integer inputs.

$$
\begin{aligned}
& g(x_1, x_2, \ldots, x_j) \stackrel{\text{def}}{\Leftarrow} \\
& \quad \text{if} \quad j = 2 \quad \text{then} \quad x_2 \\
& \quad\quad \text{else if} \quad x_1 = x_2 + 1 \quad \text{then} \quad g(x_2, \ldots, x_j) \\
& \quad\quad \text{else if} \quad x_2 = x_3 + 1 \quad \text{then} \quad \max(x_3, x_j) \\
& \quad\quad\quad \text{else} \quad x_j
\end{aligned}
$$

Question 1: Can the recursive equation be satisfied?

Knuth's Challenge

Theorem 4. The function $f(x_1, x_2, \ldots, x_m)$ satisfies the m-dimensional tarai recurrence.

That is,

$$
\begin{aligned}
& f(x_1, x_2, \ldots, x_m) = \\
& \quad \text{if } \; x_1 \le x_2 \; \text{ then } \; x_2 \\
& \quad\quad \text{else } \; f(\; f(x_1 - 1, x_2, \ldots, x_m), \\
& \quad\quad\quad\quad f(x_2 - 1, x_3, \ldots, x_m, x_1), \\
& \quad\quad\quad\quad\quad \vdots \\
& \quad\quad\quad\quad f(x_m - 1, x_1, \ldots, x_{m-1}) \;).
\end{aligned}
$$

Open Problem 4. Prove Theorem 4 by computer.

Question 1: Can the recursive equation be satisfied?

References

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Selected Papers on the Analysis of Algorithms,
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Question 1: Can the recursive equation be satisfied?

Bailey's Counterexample, when $m = 6$

Left side

$$f(8,6,4,3,1,2) = g(8,6,4,3,1,2) = 2$$

Right side *inner* evaluations

$$\begin{aligned}
f(7,6,4,3,1,2) &= g(7,6,4,3,1,2) \\
&= g(6,4,3,1,2) \\
&= \max(3,2) = 3 \\
f(5,4,3,1,2,8) &= g(5,4,3,1,2) \\
&= g(4,3,1,2) \\
&= g(3,1,2) = 2 \\
f(3,3,1,2,8,6) &= g(3,3) = 3 \\
f(2,1,2,8,6,4) &= g(2,1,2) = g(1,2) = 2 \\
f(0,2,8,6,4,3) &= g(0,2) = 2 \\
f(1,8,6,4,3,1) &= g(1,8) = 8
\end{aligned}$$

Right side outer evaluation

$$f(3,2,3,2,2,8) = g(3,2,3) = g(2,3) = 3$$

Question 1: Can the recursive equation be satisfied?

Knuth's Reaction to Counterexample

Date: Wed, 16 Feb 2000 13:42:50 -0800 (PST)

To: cowles
Subject: note from Prof Knuth
Dear Dr Cowles,

Wow!
or maybe I should say Ow!
or Whoa!
or Woe!
Thanks again for another crucial news flash.

. . .

This is certainly a great way to make me believe in mechanical verification. I presume that the case m=5 will still hold up to scrutiny by ACL2 . . .

Question 1: Can the recursive equation be satisfied?

Bailey Proposes a Correction

$$f(x_1, x_2, \ldots, x_m) \stackrel{\text{def}}{\Leftarrow}$$
$$\begin{array}{l} \text{if} \quad (\exists k < m)(x_1 > x_2 > \cdots > x_k \le x_{k+1}) \\ \qquad \text{then} \quad g_b(x_1, x_2, \ldots, x_{k+1}) \\ \qquad \text{else} \quad x_1 \end{array}$$

Here the "function" g_b, like g, takes a variable number (at least two) of integer inputs.

$$g_b(x_1, x_2, \ldots, x_j) \stackrel{\text{def}}{\Leftarrow}$$
$$\begin{array}{l} \text{if} \quad j \le 3 \quad \text{then} \quad x_j \\ \qquad \text{else if} \quad x_1 = x_2 + 1 \quad \text{or} \quad x_2 > x_3 + 1 \\ \qquad\qquad \text{then} \quad g_b(x_2, \ldots, x_j) \\ \qquad\qquad \text{else} \quad \max(x_3, x_j) \end{array}$$

Question 1: Can the recursive equation be satisfied?

Machine Verification Pending

Conjecture 1. The function $f(x_1, x_2, \ldots, x_m)$, *with* g_b *replacing* g, satisfies the m-dimensional tarai recurrence.

That is,

$$
\begin{aligned}
&f(x_1, x_2, \ldots, x_m) = \\
&\quad \text{if} \;\; x_1 \le x_2 \;\; \text{then} \;\; x_2 \\
&\qquad \text{else} \;\; f(\; f(x_1 - 1, x_2, \ldots, x_m), \\
&\qquad\qquad\quad f(x_2 - 1, x_3, \ldots, x_m, x_1), \\
&\qquad\qquad\qquad \vdots \\
&\qquad\qquad\quad f(x_m - 1, x_1, \ldots, x_{m-1}) \;).
\end{aligned}
$$

Proof checked by hand.

Question 2: Does the recursion terminate?

May Depend on Evaluation Rule

Knuth points out,
"... a call-by-need technique *will* always terminate when applied to the recursive equation for $t(x_1, \ldots, x_m)$.

If $x_1 > x_2 > \cdots > x_k \leq x_{k+1}$, the values $y_i = t(x_i - 1, x_{i+1}, \ldots, x_{i-1})$ need be expanded only for $1 \leq i \leq k+1$, and this will be sufficient to determine the value of $t(y_1, \ldots, y_m) = t(x_1, \ldots, x_m)$ in a finite number of steps."

Knuth's argument depends on the faulty proof given for **Theorem 4.**

Question 2: Does the recursion terminate?

Machine Verification Pending

Conjecture 2. The recursion for computing $t(x_1, \ldots, x_m)$ always terminates using the following version of Knuth's call-by-need.

If $x_1 > x_2 > \cdots > x_k \leq x_{k+1}$, it is sufficient to expand the values $y_i = t(x_i - 1, x_{i+1}, \ldots, x_{i-1})$ only for $1 \leq i \leq k$, to determine the value of $t(y_1, \ldots, y_m) = t(x_1, \ldots, x_m)$.

Note the change from $k + 1$ to k in this range for i.

Question 2: Does the recursion terminate?

This is an Interesting Question!

Knuth continues,
"Therefore we come to a final question, which will perhaps prove to be the most interesting aspect of the present investigation, particularly if it has a negative answer.

. . . If so, the tarai recurrence would be an extremely interesting example to include in *all* textbooks about recursion."

Open Problem 5. Does the m-dimensional tarai recursive equation define a total function, for all $m \geq 3$, if it is expanded fully (without call-by-need)?

Question 2: Does the recursion terminate?

An Interesting Example – of What?

Negative answer to **Open Problem 5:**

The m-dimensional tarai function shows that LISP's evaluation rules do *not* always compute *least fixed-points*.

Lisp's evaluation rules:

- innermost first, left to right
- call by value

Simpler examples are known.

Question 2: Does the recursion terminate?

A Simpler Example

LISP's evaluation rules do *not* always compute *least fixed-points*.

$$L(x,y) \stackrel{\text{def}}{\Leftarrow} \text{if } x = 0 \text{ then } 0 \text{ else } L(x-1, L(x,y)).$$

Only one function, over the nonnegative integers, satisfies the recursive equation.

$$L(x,y) \stackrel{\text{def}}{\Leftarrow} 0$$

Recursion does *not* terminate.
Assuming left-to-right, innermost-first evaluation order.

Question 2: Does the recursion terminate?

A Counterexample, when $m = 4$

$$
\begin{aligned}
t(3,2,1,5) &= t(\ t(2,2,1,5), \\
&\qquad t(1,1,5,3), \\
&\qquad t(0,5,3,2), \\
&\qquad t(4,3,2,1)\ ) \\
&= t(2,1,5,t(4,3,2,1)) \\
&= t(2,1,5,t(\ t(3,3,2,1), \\
&\qquad\qquad\qquad t(2,2,1,4), \\
&\qquad\qquad\qquad t(1,1,4,3), \\
&\qquad\qquad\qquad t(0,4,3,2)\ )) \\
&= t(2,1,5,t(3,2,1,4)) \\
&= t(2,1,5,t(\ t(2,2,1,4), \\
&\qquad\qquad\qquad t(1,1,4,3), \\
&\qquad\qquad\qquad t(0,4,3,2), \\
&\qquad\qquad\qquad t(3,3,2,1)\ )) \\
&= t(2,1,5,t(2,1,4,3))
\end{aligned}
$$

Question 2: Does the recursion terminate?

Counterexample continued

From last line on previous slide:

$$
\begin{aligned}
t(3,2,1,5) &= t(2,1,5,t(2,1,4,3)) \\
&= t(2,1,5,t(\ t(1,1,4,3), \\
&\qquad\qquad\qquad t(0,4,3,2), \\
&\qquad\qquad\qquad t(3,3,2,1), \\
&\qquad\qquad\qquad t(2,2,1,4)\ )) \\
&= t(2,1,5,t(1,4,3,2)) \\
&= t(2,1,5,4) \\
&= t(\ t(1,1,5,4), \\
&\qquad t(0,5,4,2), \\
&\qquad t(4,4,2,1), \\
&\qquad t(3,2,1,5)\ )
\end{aligned}
$$

Initial inputs repeated.

Progress Using ACL2

Applying ACL2, in an inelegant way with brute force, verifies the following:

- For $2 \leq m \leq 7$, Bailey's function f of **Conjecture 1** satisfies the m-dimensional tarai recurrence.
  Thus ACL2 verifies **Conjecture 1** for $2 \leq m \leq 7$.

- For $2 \leq m \leq 5$, Knuth's version of f computes the same values as Bailey's version of f given in **Conjecture 1**.

Together, these items finish the mechanical verification that Knuth's f satisfies the recursive equation, for $m = 5$ (as well as for $2 \leq m \leq 4$).

Progress Using ACL2

Version 2.4 Linear Arithmetic Misbehavior

Fixed in Version 2.5.

Note the (INTEGERP FORTH).

```
(THM
 (IMPLIES (AND (INTEGERP THIRD)
               (INTEGERP FORTH)
               (INTEGERP FIFTH)
               (< FORTH THIRD)
               (< FIFTH FORTH)
               (<= (+ -1 THIRD) FORTH)
               (<= (+ -1 FORTH) FIFTH)
               ...

          (EQUAL FIRST
                 (GB (LIST (+ -1 FIRST)
                           SECOND))))
 ... )
```

Progress Using ACL2

Version 2.4 Linear Arithmetic Misbehavior

Now there is no (INTEGERP FORTH).

```
This simplifies, using linear arithmetic ...

Goal'
(IMPLIES (AND (INTEGERP (+ 1 FORTH))
              (INTEGERP (+ 1 FIFTH))
              (INTEGERP FIFTH)
              (< (+ 1 FORTH)(+ -1 FIRST))
              (< (+ 1 FIFTH)(+ 1 FORTH))
              (< (+ 1 FORTH)(+ -1 -1 FIRST))
              (<= (+ -1 1 FORTH)(+ 1 FIFTH))
              ...

         (EQUAL FIRST
                (GB (LIST (+ -1 FIRST)
                          (+ -1 FIRST))))).
```

Progress Using ACL2

Version 2.4 Linear Arithmetic Misbehavior

```
This forcibly simplifies, using linear
arithmetic ...
But simplification reduces this to T ...
q.e.d. (given one forced hypothesis)

Modulo the following forced goal ...
[1]Goal, below, will focus on
(ACL2-NUMBERP FORTH),
which was forced in
 Goal', above,
  by the linearization of
  (EQUAL FORTH (+ 1 FIFTH)).

We now undertake Forcing Round 1.
[1]Goal
(ACL2-NUMBERP FORTH).
******** FAILED ********
```

Progress Using ACL2

ACL2 verifies

- For $2 \leq m \leq 7$, Bailey's function f of **Conjecture 1** satisfies the m-dimensional *restricted* tarai recurrence:

$$
\begin{aligned}
& f(x_1, x_2, \ldots, x_m) = \\
& \quad \text{if } \; x_1 \leq x_2 \; \text{ then } \; x_2 \\
& \qquad \text{else } \; f(\; f(x_1 - 1, x_2, \ldots, x_m), \\
& \qquad\qquad\qquad f(x_2 - 1, x_3, \ldots, x_m, x_1), \\
& \qquad\qquad\qquad\qquad \vdots \\
& \qquad\qquad\qquad f(x_k - 1, x_{k'}, \ldots, x_{k-1}) \;).
\end{aligned}
$$

- Note k and k' in the last line above.
  k satisfies $x_1 > x_2 > \cdots > x_k \leq x_{k'}$.
  $k' = (k+1) \text{ mod}' \, m$.
  $i \text{ mod}' \, m = j \in \{1, \ldots, m\} \ni i \equiv j \text{ mod } m$.

- f becomes a "function" with a variable number (at least two) of integer inputs.

Progress Using ACL2

Coping with a variable number of inputs

- The "functions" g, g_b, and the *restricted* tarai recurrence take a variable number of inputs

- Lisp provides an obvious implementation of functions with a variable number of inputs:

  Form the inputs into a list and use that list as the single input to the function.

Progress Using ACL2

ACL2 verifies

- For $2 \le m \le 7$, Bailey's function f of **Conjecture 1** is the *unique* total function on the integers that satisfies the m-dimensional tarai recurrence.

- For $2 \le m \le 7$, Bailey's function f of **Conjecture 1** is the *unique* total function on the integers that satisfies the m-dimensional *restricted* tarai recurrence.

- For $2 \le m \le 7$, the recursive calls on the right side of the m-dimensional *restricted* tarai recurrence, always terminate. Thus ACL2 verifies **Conjecture 2** for $2 \le m \le 7$.

Progress Using ACL2

Use `Encapsulate` to consistently axiomatize four functions `tarai`, `tarai-lst`, `rTarai`, and `rTarai-lst` so that for $2 \le m \le 7$

- `tarai` is a total function that satisfies the m-dimensional tarai recurrence.
- `tarai` returns an integer whenever the input is a list of integers of length 2 or more.
- `rTarai` is a total function that satisfies the m-dimensional *restricted* tarai recurrence.

The axioms specifically restrict their validity to input lists of lengths 2–7.

Bailey's f is used as the witness,

Progress Using ACL2

- *Uniqueness* is proved by cases.

  One case for each list length from 2–7.

  Induction is used to prove each case.

- The measure of lists of integers $(x_1, x_2, \ldots, x_m)$, used for the induction:

  Based on the lexicographical ordering on pairs $(k, \mathtt{nfix}(x_1 - x_2))$.

  Here k is the integer such that $x_1 > x_2 > \cdots > x_k \leq x_{k'}$.

  k' equals $(k+1)$ mod$'$ m.

- Use same measure to demonstrate termination of the *restricted* tarai recurrence.

Current Work

Use ACL2 to elegantly prove for all integers $m \geq 2$:

- Bailey's f *uniquely* satisfies the m-dimensional tarai and *restricted* tarai recurrences.

- The recursion always terminates for the *restricted* tarai recurrence.