Solving $\triangle = \square$

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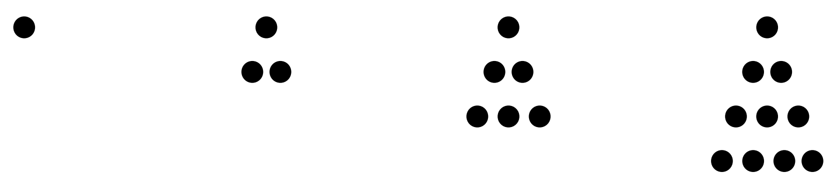
For positive integer n,
the **triangular number**, Δ_n,
is defined by

$$\begin{aligned} \Delta_n &= \sum_{i=1}^{n} i = 1 + 2 + \cdots + (n-1) + n \\ &= \frac{n \cdot (n+1)}{2}. \end{aligned}$$

The first 4 triangular numbers:



$\Delta_1 = 1 \quad \Delta_2 = 3 \quad \Delta_3 = 6 \quad \Delta_4 = 10$

Problem. Find triangular numbers that are also squares.
That is, find positive integers, n and k, such that

$$\frac{n \cdot (n+1)}{2} = k^2$$

or

$$n \cdot (n+1) = 2 \cdot k^2.$$

Clearly $\Delta_1 = 1$ is a square.

Are there other square triangular numbers?

Are there infinitely many square triangular numbers?

Answer these questions.

The answers are formally verified using ACL2.

Reformulate the Problem

Transform the original problem into one that has been studied for a long time and has a well understood solution.

Lemma. If n and k are positive integers such that

$$n \cdot (n+1) = 2 \cdot k^2,$$

then $x = 2 \cdot n + 1$ and $y = 2 \cdot k$ are positive integers such that

$$x^2 - 2 \cdot y^2 = 1.$$

Moreover, $x \geq 3$ is odd and $y \geq 2$ is even.

Lemma. If x and y are positive integers such that

$$x^2 - 2 \cdot y^2 = 1,$$

then $n = \frac{x-1}{2}$ and $k = \frac{y}{2}$ are positive integers such that

$$n \cdot (n+1) = 2 \cdot k^2.$$

The proofs use elementary algebra.

Pell's Equation

For squarefree positive integers D, the equation

$$x^2 - D \cdot y^2 = 1$$

is usually called Pell's equation.

John Pell (1610–1685) is an English Mathematician.

He made **no** contribution to the study of this equation!

Original problem: Find positive integers n and k such that

$$n \cdot (n+1) = 2 \cdot k^2$$

Equivalent Pell Eqn: Find positive integers x and y such that

$$x^2 - 2 \cdot y^2 = 1.$$

Original problem: Find positive integers n and k such that

$$n \cdot (n+1) = 2 \cdot k^2$$

Equivalent Pell Eqn: Find positive integers x and y such that

$$x^2 - 2 \cdot y^2 = 1.$$

The obvious solution, $n = 1, k = 1$, to original problem corresponds to the solution, $x = 3, y = 2$, of the Pell equation.

Generate Many Solutions

Construct new solutions to our Pell equation from known solutions:

Lemma. If a, b, c, and d are positive integers such that

$$a^2 - 2 \cdot b^2 = 1,$$

and

$$c^2 - 2 \cdot d^2 = 1,$$

then $x = a \cdot c + 2 \cdot b \cdot d$ and $y = a \cdot d + b \cdot c$ are positive integers such that

$$x^2 - 2 \cdot y^2 = 1.$$

The proof uses elementary algebra.

Lemma from previous slide:

Lemma. If a, b, c, and d are positive integers such that

$$a^2 - 2 \cdot b^2 = 1,$$

and

$$c^2 - 2 \cdot d^2 = 1,$$

then $x = a \cdot c + 2 \cdot b \cdot d$ and $y = a \cdot d + b \cdot c$ are positive integers such that

$$x^2 - 2 \cdot y^2 = 1.$$

In the above lemma, the two given solutions, a, b and c, d, need not be distinct.

That is, using $a = c$ and $b = d$ in the construction of x, y yields a new solution.

Thus, starting with one known solution, $a = 3, b = 2$, many other solutions can be generated:

Definition 1. For each positive integer, j, recursively define (x_j, y_j) by

$$\begin{aligned}(x_1, y_1) &= (3, 2)\\ (x_{j+1}, y_{j+1}) &= (x_1 \cdot x_j + 2 \cdot y_1 \cdot y_j,\\ &\qquad x_1 \cdot y_j + x_j \cdot y_1)\end{aligned}$$

Examples.

j	1	2	3	4	5	6
x_j	3	17	99	577	3363	19601
y_j	2	12	70	408	2378	13860

j	7	8	9
x_j	114243	665857	3880899
y_j	80782	470832	2744210

This theorem is proved by mathematical induction on j.

Theorem 1. For each positive integer j,

$$x_j^2 - 2 \cdot y_j^2 = 1.$$

No Other Solutions

Definition 1. For each positive integer, j, recursively define (x_j, y_j) by

$$\begin{aligned}(x_1, y_1) &= (3, 2)\\ (x_{j+1}, y_{j+1}) &= (x_1 \cdot x_j + 2 \cdot y_1 \cdot y_j,\\ &\quad x_1 \cdot y_j + x_j \cdot y_1)\end{aligned}$$

Solutions given by **Definition 1** are all the positive integer solutions:

Theorem 2. If x and y are positive integers such that

$$x^2 - 2 \cdot y^2 = 1,$$

then for some positive integer j,

$$(x, y) = (x_j, y_j).$$

Construct another new solution, (a, b), to our Pell equation from a known solution, (x, y).

The new solution is "smaller" than the old solution in the sense that $b < y$.

Lemma. If x and y are positive integers such that $y > 2$ and

$$x^2 - 2 \cdot y^2 = 1,$$

then $a = 3 \cdot x - 4 \cdot y$ and $b = -2 \cdot x + 3 \cdot y$ are positive integers such that

$$a^2 - 2 \cdot b^2 = 1.$$

Moreover, $b < y$.

The proof uses elementary algebra.

Theorem 2. If x and y are positive integers such that

$$x^2 - 2 \cdot y^2 = 1,$$

then for some positive integer j,

$$(x, y) = (x_j, y_j).$$

Proof. By contradiction.

Choose positive integers x and y, **with y as small as possible**, such that for all positive integers j,

$$(x, y) \neq (x_j, y_j)$$

and

$$x^2 - 2 \cdot y^2 = 1.$$

Since $y \neq y_j$ for any j, then $y > 2$.

Proof cont. By **Lemma**,

$$a = 3 \cdot x - 4 \cdot y$$

and

$$b = -2 \cdot x + 3 \cdot y$$

are positive integers that satisfy

$$b < y$$

and

$$a^2 - 2 \cdot b^2 = 1.$$

Since y **is as small as possible** and $b < y$, there must be a j such that

$$(a, b) = (x_j, y_j).$$

By **Definition 1**,

$$\begin{aligned}(x_{j+1}, y_{j+1}) &= (x_1 \cdot x_j + 2 \cdot y_1 \cdot y_j, \\ &\quad\; x_1 \cdot y_j + x_j \cdot y_1) \\ &= (3 \cdot a + 2 \cdot 2 \cdot b, \\ &\quad\; 3 \cdot b + a \cdot 2) \\ &= (x, y).\end{aligned}$$

Proof cont. Here is the contradiction:

Positive integers x and y were chosen so that for all positive integers j,

$$(x, y) \neq (x_j, y_j).$$

Positive integers a and b were constructed so that there must be a j such that

$$(a, b) = (x_j, y_j)$$

and

$$(x_{j+1}, y_{j+1}) = (x, y).$$

Translate solutions of the Pell equation into solutions of the original problem.

Definition 2. For each positive integer, j, define (n_j, k_j) by

$$(n_j, k_j) = \left(\frac{x_j - 1}{2}, \frac{y_j}{2}\right).$$

Examples.

j	1	2	3	4	5	6
n_j	1	8	49	288	1681	9800
k_j	1	6	35	204	1189	6930

j	7	8	9
n_j	57121	332928	1940449
k_j	40391	235416	1372105

Theorem 3. For each positive integer j,

$$\frac{n_j \cdot (n_j + 1)}{2} = k_j^2.$$

If n and k are positive integers such that

$$\frac{n \cdot (n + 1)}{2} = k^2,$$

then for some positive integer j,

$$(n, k) = (n_j, k_j).$$

Conclusion

An algorithm, described in **Definition 1** and **Definition 2**, is presented that enumerates all pairs,

$$(n, k),$$

of positive integers such that

$$\Delta_n = k^2.$$

ACL2 is used to verify that the algorithm is correct.