Formal Methods: Practice and Pedagogy

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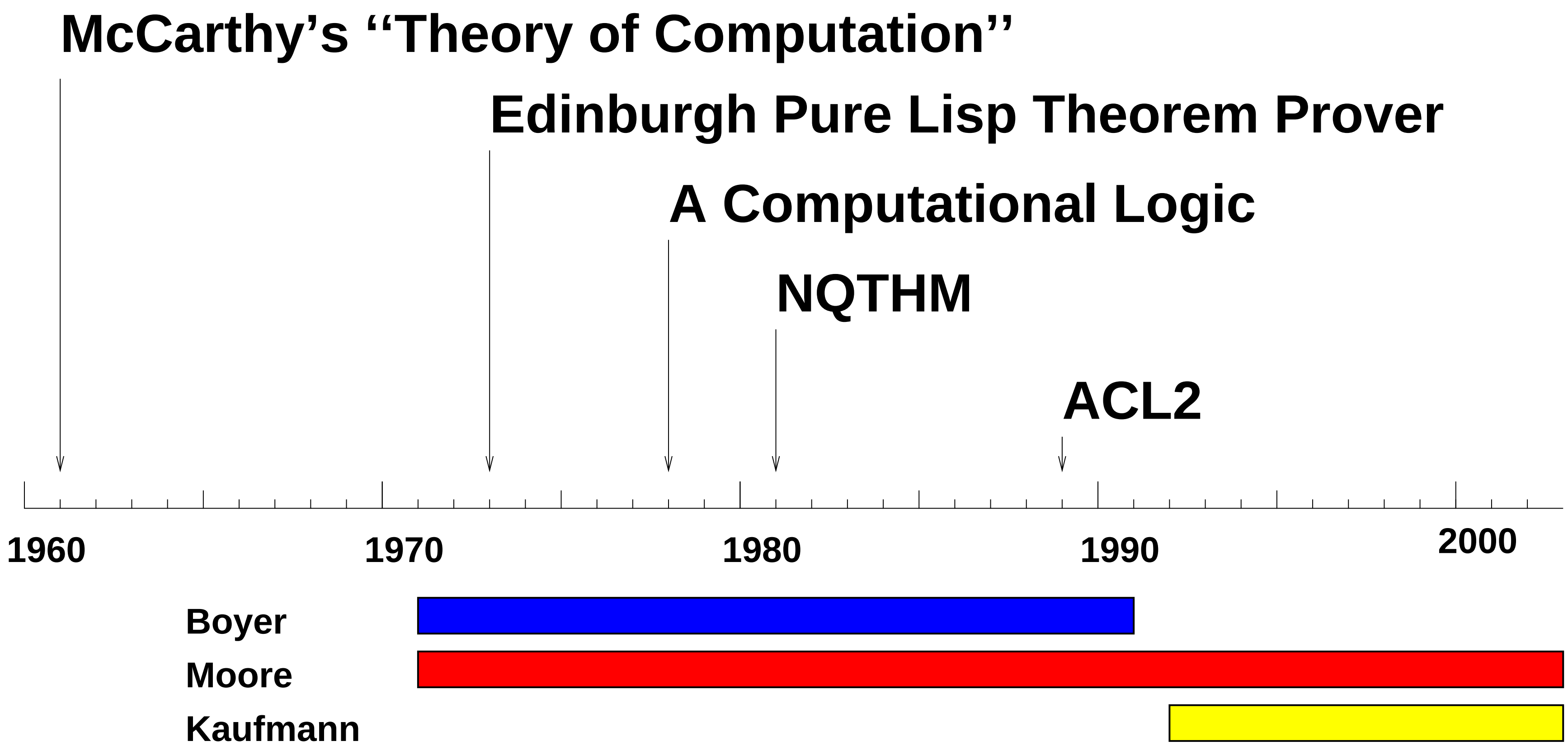
Formal Methods

Instead of debugging a program, one should prove that it meets its specifications, and this proof should be checked by a computer program.

— *John McCarthy, "A Basis for a Mathematical Theory of Computation," 1961*

Boyer-Moore Project



The Expressiveness Spectrum

Prop Calc — **Set Theory**

ACL2

BDD
zChaff

PVS
HOL
Coq

SMV
COSPAN

A Classic Challenge Theorem

Theorem: List concatenation ("append") is associative.

```
(equal (append (append a b) c)
       (append a (append b c)))
```

$\forall a \forall b$

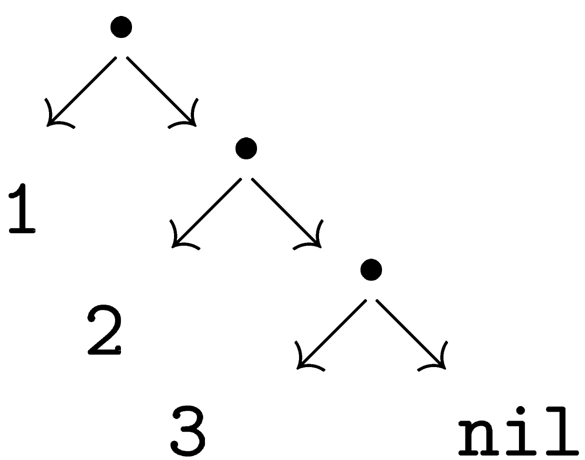
ppend(append(a,b),c)

$=$

ppend(a,append(b,c)).

Examples

```
(cons 1 (cons 2 (cons 3 nil)))
= '(1 2 3)
```




```
(append '(1 2 3) (append '(4 5 6) '(7 8 9)))
= '(1 2 3 4 5 6 7 8 9)
```

The Definition of `append`

```
(defun append (x y)
  (if (endp x)
      y
      (cons (car x)
            (append (cdr x) y))))
```

$$\texttt{(endp x)} \rightarrow \texttt{(append x y)} = \texttt{y}$$

$$\neg\texttt{(endp x)} \rightarrow \texttt{(append x y)} = \texttt{(cons (car x) (append (cdr x) y))}$$

A Few Axioms

$\texttt{t} \neq \texttt{nil}$

$\texttt{x} = \texttt{nil} \rightarrow \texttt{(if x y z)} = \texttt{z}$

$\texttt{x} \neq \texttt{nil} \rightarrow \texttt{(if x y z)} = \texttt{y}$

$\texttt{(car (cons x y))} = \texttt{x}$

$\texttt{(cdr (cons x y))} = \texttt{y}$

$\texttt{(endp nil)} = \texttt{t}$

$\texttt{(endp (cons x y))} = \texttt{nil}$

```
(equal (append (append a b) c)
       (append a (append b c)))
```

```
(equal (append (append a b) c)
       (append a (append b c)))
```

Proof: by induction on a.

```
(equal (append (append a b) c)
       (append a (append b c)))
```

Proof: by induction on a.

Base Case: (endp a).

```
(equal (append (append a b) c)
       (append a (append b c)))
```

```
(equal (append (append a b) c)
       (append a (append b c)))
```

Proof: by induction on a.

Base Case: (endp a).

```
(equal (append b c)
       (append a (append b c)))
```

```
(equal (append (append a b) c)
       (append a (append b c)))
```

```
Proof: by induction on a.
```

```
Base Case:  (endp a).
(equal (append b c)
       (append a (append b c)))
```

```
(equal (append (append a b) c)
       (append a (append b c)))
```

Proof: by induction on a.

Base Case: `(endp a)`.

```
(equal (append b c)
       (append b c))
```

```
(equal (append (append a b) c)
       (append a (append b c)))
```

Proof: by induction on a.

Base Case: (endp a).

```
(equal (append b c)
       (append b c))
```

```
(equal (append (append a b) c)
       (append a (append b c)))
```

Proof: by induction on a.

Base Case: (endp a).

T

```
(equal (append (append a b) c)
       (append a (append b c)))
```

Proof: by induction on a.

Induction Step: (not (endp a)).

```
(equal (append (append a b) c)
       (append a (append b c)))
```

```
(equal (append (append a b) c)
       (append a (append b c)))
```

Proof: by induction on a.

Induction Step: (not (endp a)).

```
(equal (append (cons (car a)
                     (append (cdr a) b)) c)
       (append a (append b c)))
```

```
(equal (append (append a b) c)
       (append a (append b c)))
```

Proof: by induction on a.

Induction Step: (not (endp a)).

```
(equal (append (cons (car a)
                     (append (cdr a) b)) c)
       (append a (append b c)))
```

```
(equal (append (append a b) c)
       (append a (append b c)))
```

```
Proof: by induction on a.
```

```
Induction Step: (not (endp a)).
(equal (cons (car a)
         (append (append (cdr a) b) c))
       (append a (append b c)))
```

```
(equal (append (append a b) c)
       (append a (append b c)))
```

Proof: by induction on a.

Induction Step: (not (endp a)).

```
(equal (cons (car a)
             (append (append (cdr a) b) c))
       (append a (append b c)))
```

```
(equal (append (append a b) c)
       (append a (append b c)))
```

Proof: by induction on a.

Induction Step: (not (endp a)).

```
(equal (cons (car a)
         (append (append (cdr a) b) c))
       (cons (car a)
         (append (cdr a) (append b c))))
```

```
(equal (append (append a b) c)
       (append a (append b c)))
```

Proof: by induction on a.

Induction Step: (not (endp a)).

```
(equal (cons (car a)
         (append (append (cdr a) b) c))
       (cons (car a)
         (append (cdr a) (append b c))))
```

```
(equal (append (append a b) c)
       (append a (append b c)))
```

```
Proof: by induction on a.
```

```
Induction Step: (not (endp a)).
(equal
         (append (append (cdr a) b) c)

         (append (cdr a) (append b c)))
```

```
(equal (append (append a b) c)
       (append a (append b c)))
```

Proof: by induction on a.

Induction Step: (not (endp a)).

```
(equal (append (append (cdr a) b) c)
       (append (cdr a) (append b c)))
```

```
(equal (append (append a b) c)
       (append a (append b c)))
```

Proof: by induction on a.

Induction Step: (not (endp a)).

```
(equal (append (append (cdr a) b) c)
       (append (cdr a) (append b c)))
```

```
(equal (append (append a b) c)
       (append a (append b c)))
```

Proof: by induction on a.

Induction Step: (not (endp a)).

T

```
(equal (append (append a b) c)
       (append a (append b c)))
```

Proof: by induction on a.

Q.E.D.

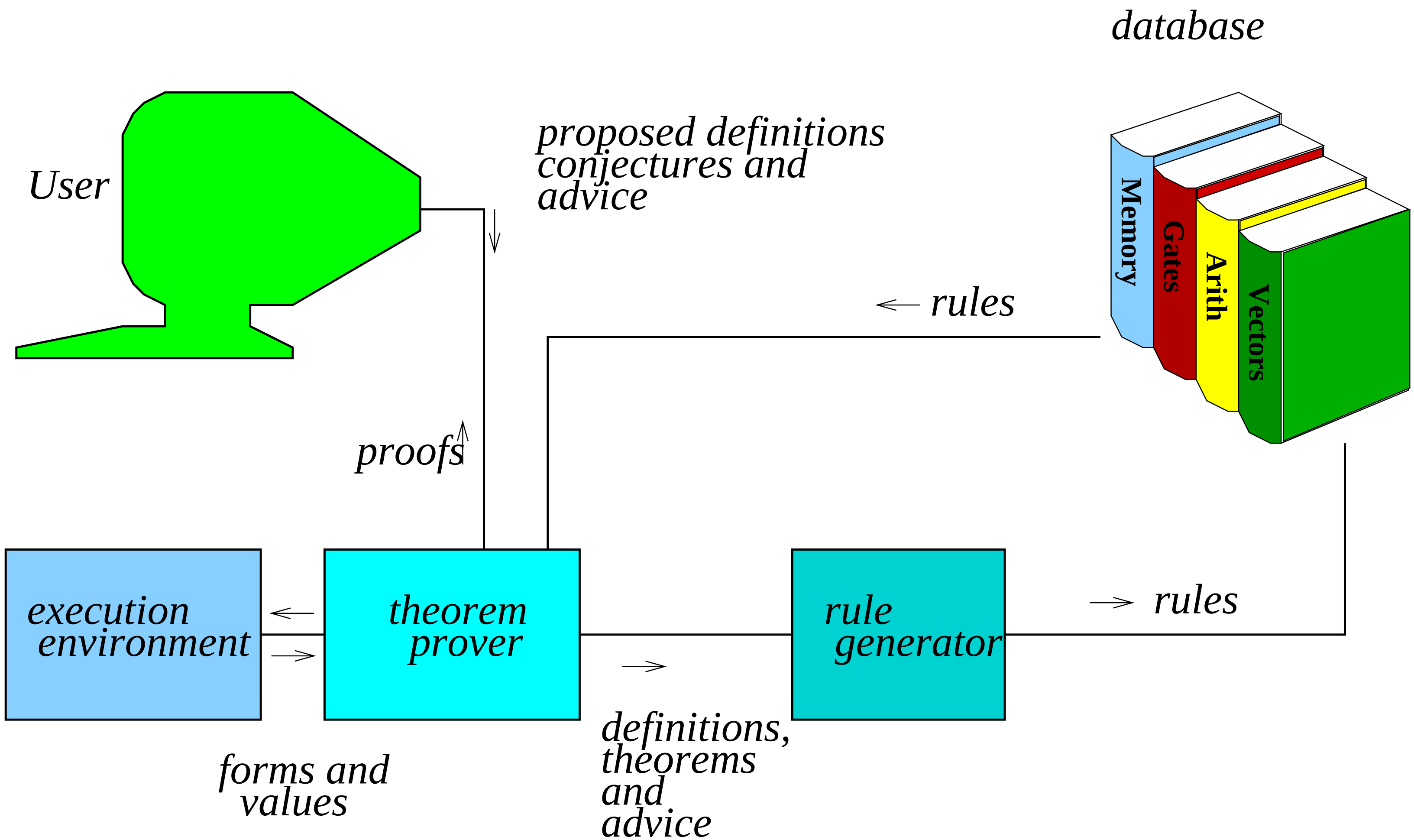
```
(defun append (x y)
  (if (endp x)
      y
      (cons (car x)
            (append (cdr x) y))))
```

Induction and recursion are duals.

Theorem

```
(equal (append (append a b) c)
       (append a (append b c)))
```

Demo 1


database
User
proposed definitions
conjectures and
advice
rules
proofs
Memory
Gates
Arith
Vectors
execution
environment
theorem
prover
rule
generator
rules
forms and
values
definitions,
theorems
and
advice

The Need for Formal Methods in Practice

An elusive circuitry error is causing a chip used in millions of computers to generate inaccurate results

— *NY Times, "Circuit Flaw Causes Pentium Chip to Miscalculate, Intel Admits," Nov 11, 1994*

Intel Corp. last week took a $475 million write-off to cover costs associated with the divide bug in the Pentium microprocessor's floating-point unit — *EE Times, Jan 23, 1995*

AMD K5 Algorithm FDIV$(p, d, mode)$

1.	sd_0	$= \mathrm{lookup}(d)$	`[exact  17  8]`
2.	d_r	$= d$	`[away   17 32]`
3.	sdd_0	$= sd_0 \times d_r$	`[away   17 32]`
4.	sd_1	$= sd_0 \times \mathrm{comp}(sd_0, 32)$	`[trunc  17 32]`
5.	sdd_1	$= sd_1 \times d_r$	`[away   17 32]`
6.	sd_2	$= sd_1 \times \mathrm{comp}(sd_1, 32)$	`[trunc  17 32]`
...	...	$= \ldots$	...
29.	q_3	$= sd_2 \times ph_3$	`[trunc  17 24]`
30.	qq_2	$= q_2 + q_3$	`[sticky 17 64]`
31.	qq_1	$= qq_2 + q_1$	`[sticky 17 64]`
32.	$fdiv$	$= qq_1 + q_0$	$mode$

Using the Reciprocal

```
        3 6.
  +       -.1 7
  +         .0 0 3 4
  +       -.0 0 0 0 6 6
     ------------------
        3 5.8 3 3 3 3 4
       ----------------
 1 2 ) 4 3 0.0 0 0 0 0 0
       4 3 2.
       ------
          -2.
          -2.0 4
          ------
            .0 4
            .0 4 0 8
            --------
          - .0 0 0 8
          - .0 0 0 7 9 2
            ------------
          - .0 0 0 0 0 8
```

Reciprocal Calculation:

$1/12 \ = \ 0.083\overline{3} \approx 0.083 = sd_2$

Quotient Digit Calculation:

$0.083 \times 430.0000 = 35.6900000 \approx 36.000000 = q_0$

$0.083 \times -2.0000 = -.1660000 \approx -.170000 = q_1$

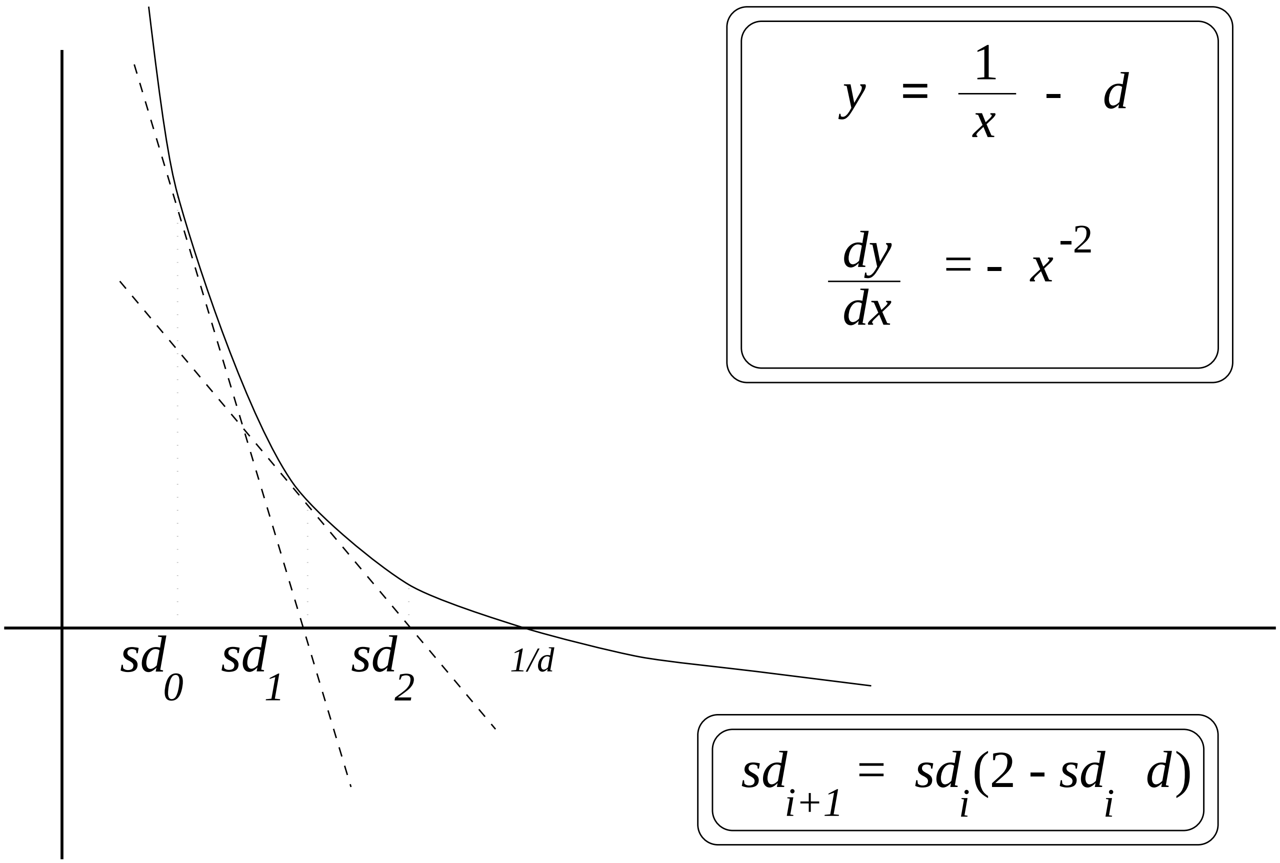
$0.083 \times .0400 = .0033200 \approx .003400 = q_2$

$0.083 \times -.0008 = -.0000664 \approx -.000067 = q_3$

Summation of Quotient Digits:

$q_0 + q_1 + q_2 + q_3 = 35.833333$

Computing the Reciprocal



$$y = \frac{1}{x} - d$$

$$\frac{dy}{dx} = -x^{-2}$$

$$sd_{i+1} = sd_i(2 - sd_i\, d)$$

top 8 bits of d	approx inverse	top 8 bits of d	approx inverse	top 8 bits of d	approx inverse	top 8 bits of d	approx inverse
1.0000000_2	0.11111111_2	1.0100000_2	0.11001100_2	1.1000000_2	0.10101010_2	1.1100000_2	0.10010010_2
1.0000001_2	0.11111101_2	1.0100001_2	0.11001011_2	1.1000001_2	0.10101001_2	1.1100001_2	0.10010001_2
1.0000010_2	0.11111011_2	1.0100010_2	0.11001010_2	1.1000010_2	0.10101000_2	1.1100010_2	0.10010001_2
1.0000011_2	0.11111001_2	1.0100011_2	0.11001000_2	1.1000011_2	0.10101000_2	1.1100011_2	0.10010000_2
1.0000100_2	0.11110111_2	1.0100100_2	0.11000111_2	1.1000100_2	0.10100111_2	1.1100100_2	0.10001111_2
1.0000101_2	0.11110101_2	1.0100101_2	0.11000110_2	1.1000101_2	0.10100110_2	1.1100101_2	0.10001111_2
1.0000110_2	0.11110100_2	1.0100110_2	0.11000101_2	1.1000110_2	0.10100101_2	1.1100110_2	0.10001110_2
1.0000111_2	0.11110010_2	1.0100111_2	0.11000100_2	1.1000111_2	0.10100100_2	1.1100111_2	0.10001110_2
1.0001000_2	0.11110000_2	1.0101000_2	0.11000010_2	1.1001000_2	0.10100011_2	1.1101000_2	0.10001101_2
1.0001001_2	0.11101110_2	1.0101001_2	0.11000001_2	1.1001001_2	0.10100011_2	1.1101001_2	0.10001100_2
1.0001010_2	0.11101101_2	1.0101010_2	0.11000000_2	1.1001010_2	0.10100010_2	1.1101010_2	0.10001100_2
...	...	...	...	...	...	...	...
1.0010110_2	0.11011010_2	1.0110110_2	0.10110100_2	1.1010110_2	0.10011001_2	1.1110110_2	0.10000101_2
1.0010111_2	0.11011000_2	1.0110111_2	0.10110011_2	1.1010111_2	0.10011000_2	1.1110111_2	0.10000100_2
1.0011000_2	0.11010111_2	1.0111000_2	0.10110010_2	1.1011000_2	0.10010111_2	1.1111000_2	0.10000100_2
1.0011001_2	0.11010101_2	1.0111001_2	0.10110001_2	1.1011001_2	0.10010111_2	1.1111001_2	0.10000011_2
1.0011010_2	0.11010100_2	1.0111010_2	0.10110000_2	1.1011010_2	0.10010110_2	1.1111010_2	0.10000011_2
1.0011011_2	0.11010011_2	1.0111011_2	0.10101111_2	1.1011011_2	0.10010101_2	1.1111011_2	0.10000010_2
1.0011100_2	0.11010001_2	1.0111100_2	0.10101110_2	1.1011100_2	0.10010101_2	1.1111100_2	0.10000010_2
1.0011101_2	0.11010000_2	1.0111101_2	0.10101101_2	1.1011101_2	0.10010100_2	1.1111101_2	0.10000001_2
1.0011110_2	0.11001111_2	1.0111110_2	0.10101100_2	1.1011110_2	0.10010011_2	1.1111110_2	0.10000001_2
1.0011111_2	0.11001101_2	1.0111111_2	0.10101011_2	1.1011111_2	0.10010011_2	1.1111111_2	0.10000000_2

The Formal Model of the Code

```
(defun FDIV (p d mode)
 (let*
   (sd0  (eround (lookup d)                '(exact  17  8)))
   (dr   (eround d                         '(away   17 32)))
   (sdd0 (eround (* sd0 dr)                '(away   17 32)))
   (sd1  (eround (* sd0 (comp sdd0 32)) '(trunc  17 32)))
   (sdd1 (eround (* sd1 dr)                '(away   17 32)))
   (sd2  (eround (* sd1 (comp sdd1 32)) '(trunc  17 32)))
   ...
   (qq2  (eround (+ q2 q3)                 '(sticky 17 64)))
   (qq1  (eround (+ qq2 q1)                '(sticky 17 64)))
   (fdiv (round   (+ qq1 q0)               mode)))
  (or (first-error sd0 dr sdd0 sd1 sdd1 ... fdiv)
      fdiv)))
```

IEEE 754 Floating Point Standard

Elementary operations are to be performed as though the infinitely precise (standard mathematical) operation were performed and then the result rounded to the indicated precision.

The K5 FDIV Theorem (1200 lemmas)

```
(defthm FDIV-divides
  (implies (and (floating-point-numberp p 15 64)
                (floating-point-numberp d 15 64)
                (not (equal d 0))
                (rounding-modep mode))
           (equal (FDIV p d mode)
                  (round  (/ p d) mode))))
```

(by Moore, Lynch and Kaufmann, in 1995, *before the K5 was fabricated*)

A Lemma from the FP Books

Lemma 7.3.2 (Sticky Plus)
Let x be a non-0 rational that fits in $n > 0$ bits, which is to say $\mathrm{trunc}(x, n) = x$. Let y be a rational whose exponent is at least two smaller than that of x, $1 + \mathrm{e}_y < \mathrm{e}_x$. Let k be a positive integer such that $n + \mathrm{e}_y - \mathrm{e}_x < k$.

$$\overbrace{x\,\overbrace{xxxxxxxxxxxxxxxx}^{\mathrm{e}_x}.xxxxxxxxxxxx}^{n\ bits}$$

$$\underbrace{y\,\underbrace{yyyyyyyyyyyy}_{\mathrm{e}_y}.yyyyyyyyyyyyyy\ yyy}_{k\ bits}\ldots$$

Then $\mathrm{sticky}(x + y, n) = \mathrm{sticky}(x + \mathrm{sticky}(y, k), n)$.

AMD Athlon 1997

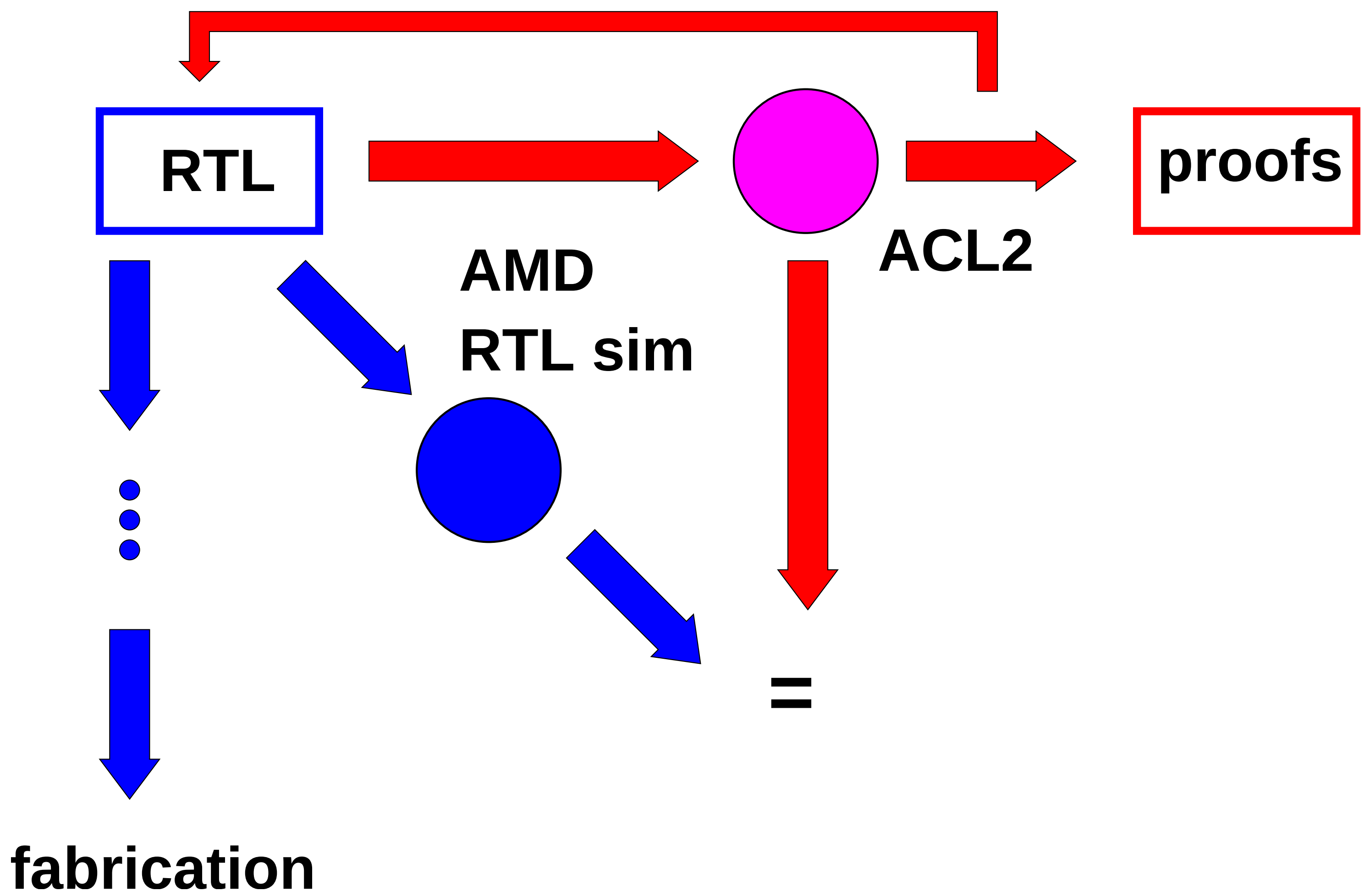
All elementary floating-point operations, `FADD`, `FSUB`, `FMUL`, `FDIV`, and `FSQRT`, on the AMD Athlon were

- specified in ACL2 to be IEEE compliant,
- proved to meet their specifications, and
- the proofs were checked mechanically.

AMD Athlon FMUL

```
module FMUL;  // sanitized from AMD Athlon(TM)
              // by David Russinoff and Art Flatau
//***************************************************
// Declarations
//***************************************************
//Precision and rounding control:
`define SNG   1'b0     // single precision
`define DBL   1'b1     // double precision
`define NRE   2'b00    // round to nearest
`define NEG   2'b01    // round to minus infinity
`define POS   2'b10    // round to plus infinity
```

```
//Parameters:
input x[79:0];            //first operand
input y[79:0];            //second operand
input rc[1:0];            //rounding control
input pc;                 //precision control
output z[79:0];           //rounded product
//************************************************
// First Cycle
//************************************************
//Operand fields:
sgnx = x[79]; sgny = y[79];
expx[14:0] = x[78:64]; expy[14:0] = y[78:64];
```


RTL
proofs
ACL2
AMD
RTL sim
=
fabrication

The Athlon FMUL Theorem

```
(let ((ideal (rnd (* (hat x) (hat y))
                  (mode rc)
                  (precision pc)))
      (z (fmul x y rc pc)))
  (implies (and (normal-encoding-p x (extfmt))
                (normal-encoding-p y (extfmt))
                (member rc (list 0 1 2 3))
                (member pc (list 0 1))
                (repp ideal (extfmt)))
           (and (normal-encoding-p z (extfmt))
                (= (hat z) ideal))))
```

The ACL2 proofs uncovered bugs that had remained hidden through hundreds of millions of test cases in RTL simulators.

The bugs were fixed and the new RTL verified *before* the Athlon was fabricated.

This work was done primarily by David Russinoff and Art Flatau, of AMD.

Other Work at AMD

AMD is using ACL2 to reason about multi-processor implementations, at the algorithm level and close to the RTL level.

They have proved a progress theorem about a model hand-derived from the RTL.

They have proved correctness at the algorithm level of a mechanism related to speculative reads.

New bugs (which were undetected after simulation) have been found and fixed before tapeout.

Other Commercial Work

- FDIV on AMD K5 (Moore-Kaufmann-Lynch)
- AMD Athlon floating point (Russinoff-Flatau)
- Motorola 68020 and Berkeley C String Library (Yu)
- $\ldots$

Motorola 68020 and the C String Library

```
/* copy char from[] to to[] */
char *
strcpy(to, from)
   register char *to, *from;
{ char *save = to;
   for (; *to = *from; ++from, ++to);
   return(save);
}
gcc -o ...
```

```
0x2558 <strcpy>:    linkw fp,#0
0x255c <strcpy+4>:  moveal fp@(8),a0
0x2560 <strcpy+8>:  moveal fp@(12),a1
0x2564 <strcpy+12>: movel a0,d1
0x2566 <strcpy+14>: bra 0x256c <strcpy+20>
0x2568 <strcpy+16>: adqw #1,a1
0x256a <strcpy+18>: adqw #1,a0
x256c <strcpy+20>: moveb a1@,d0
```

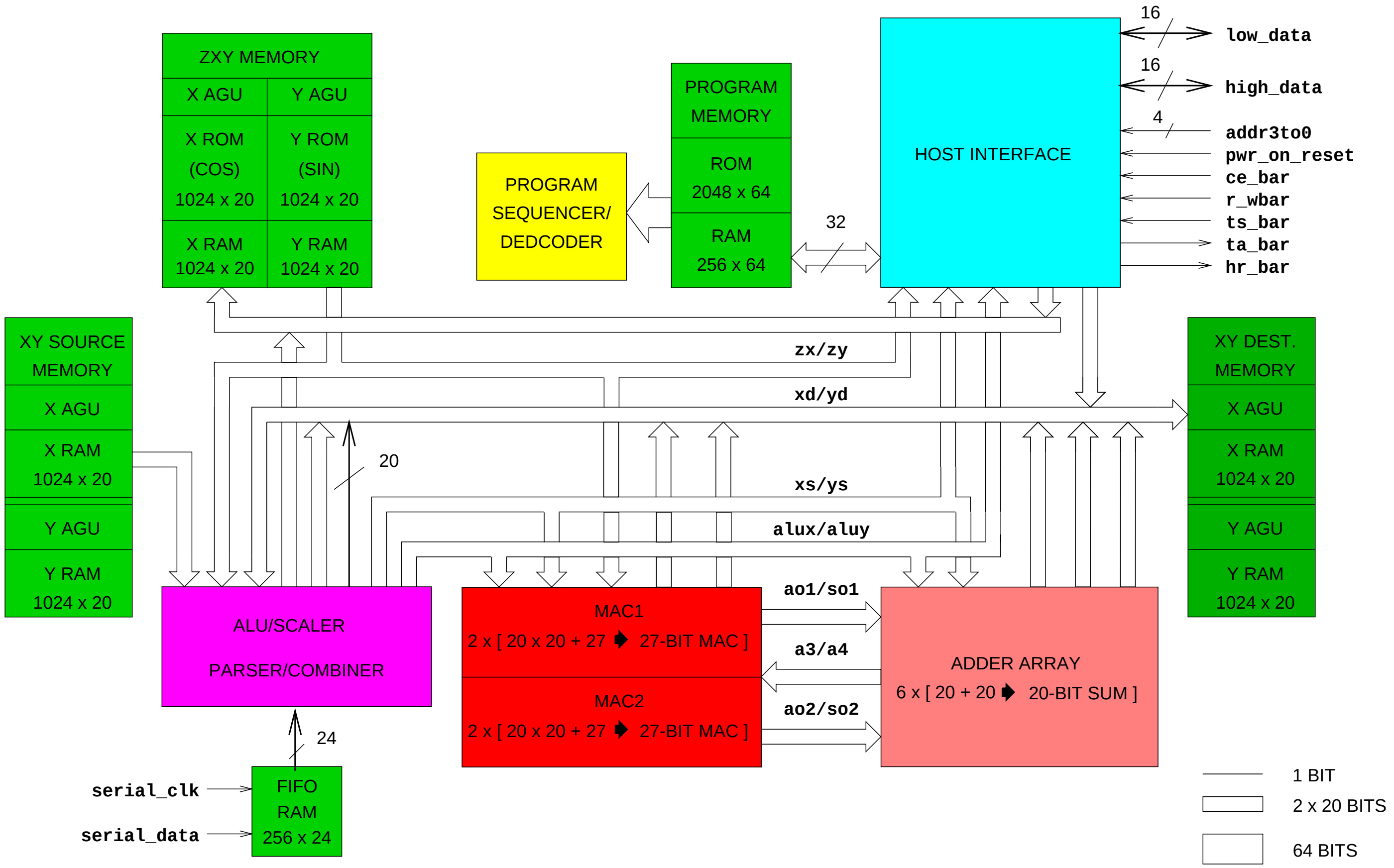
Other Commercial Work

- Motorola 68020 and Berkeley C String Library (Yu)

- Motorola CAP DSP (Brock)

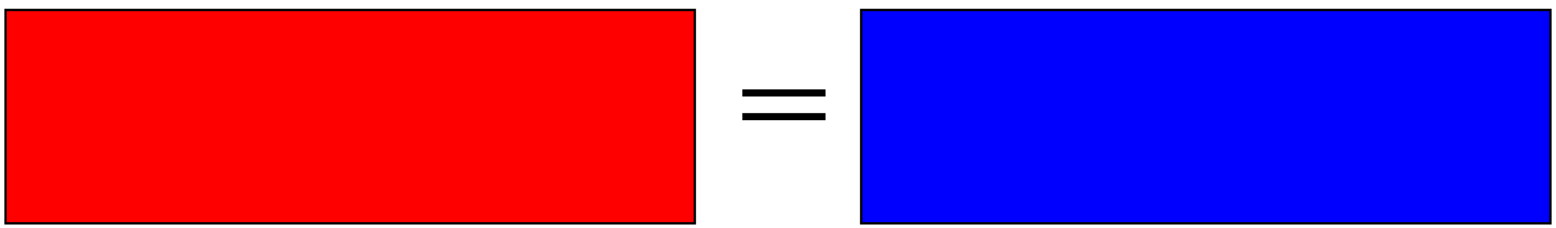
- $\therefore$

Motorola CAP DSP



ROM containing 50 microcoded DSP algorithms

Pipelined microarchitecture **Sequential microcode ISA**



(if no hazards)

Other Commercial Work

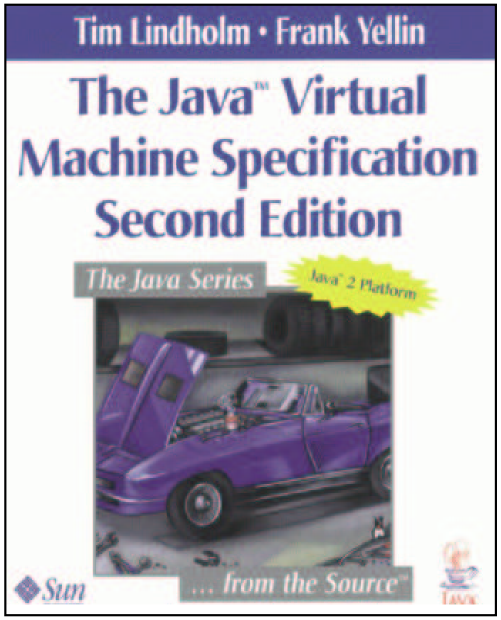
- ...
- Motorola CAP DSP (Brock)
- IBM 4758 secure co-processor (Austel)
- Union Switch and Signal safety-critical checker (Bertolli)
- $\ddots$

The security model was formalized in ACL2 and certain properties were proved to obtain FIPS 140-1 Level Four certification.

Other Commercial Work

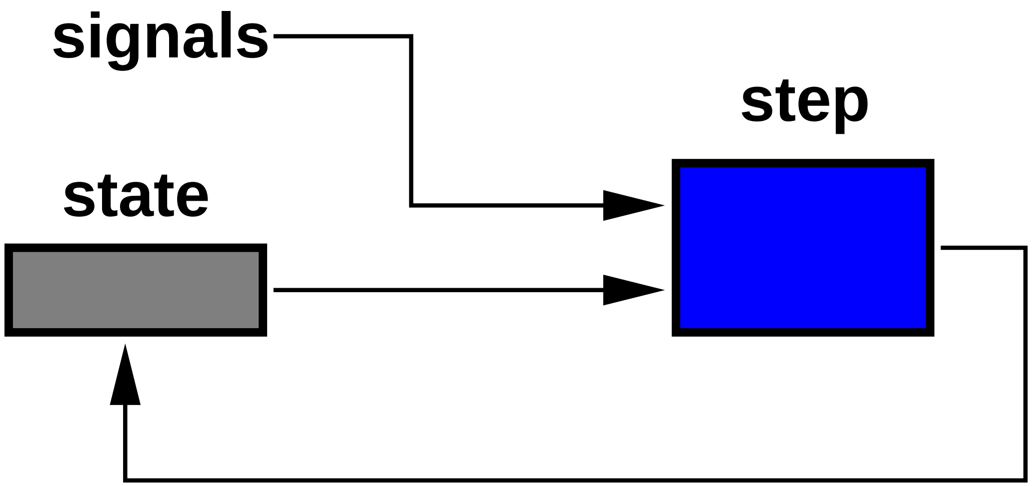
- $\iddots$
- IBM 4758 secure co-processor (Austel)
- Union Switch and Signal safety-critical checker (Bertolli)
- Rockwell Collins / aJile Systems JEM1 (Hardin-Greve-Wilding)
- Java and the JVM (UT Austin with Sun and others)

Java and the JVM



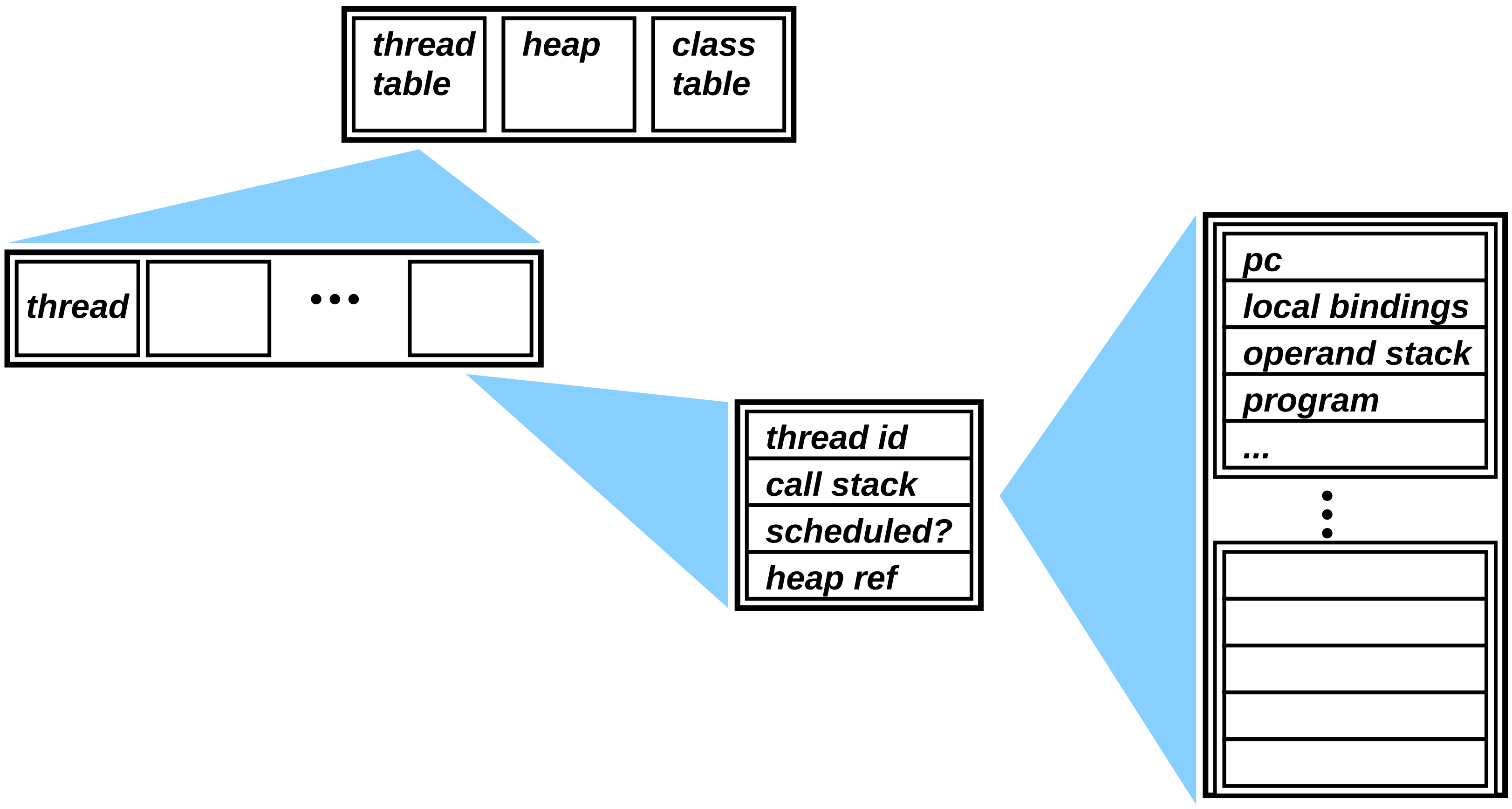

```
; JVM in ACL2


(defun make-state (tt hp ct)
   ...)

(defun step (th s)
   ...)

(defun run (sched s)
  (if (endp sched)
      s
    (run
      (cdr sched)
      (step (car sched) s))))
```

Demo 2




```
(defun run (signals state)
  (if (endp signals)
      state
      (run (cdr signals)
           (step (car signals) state))))
```

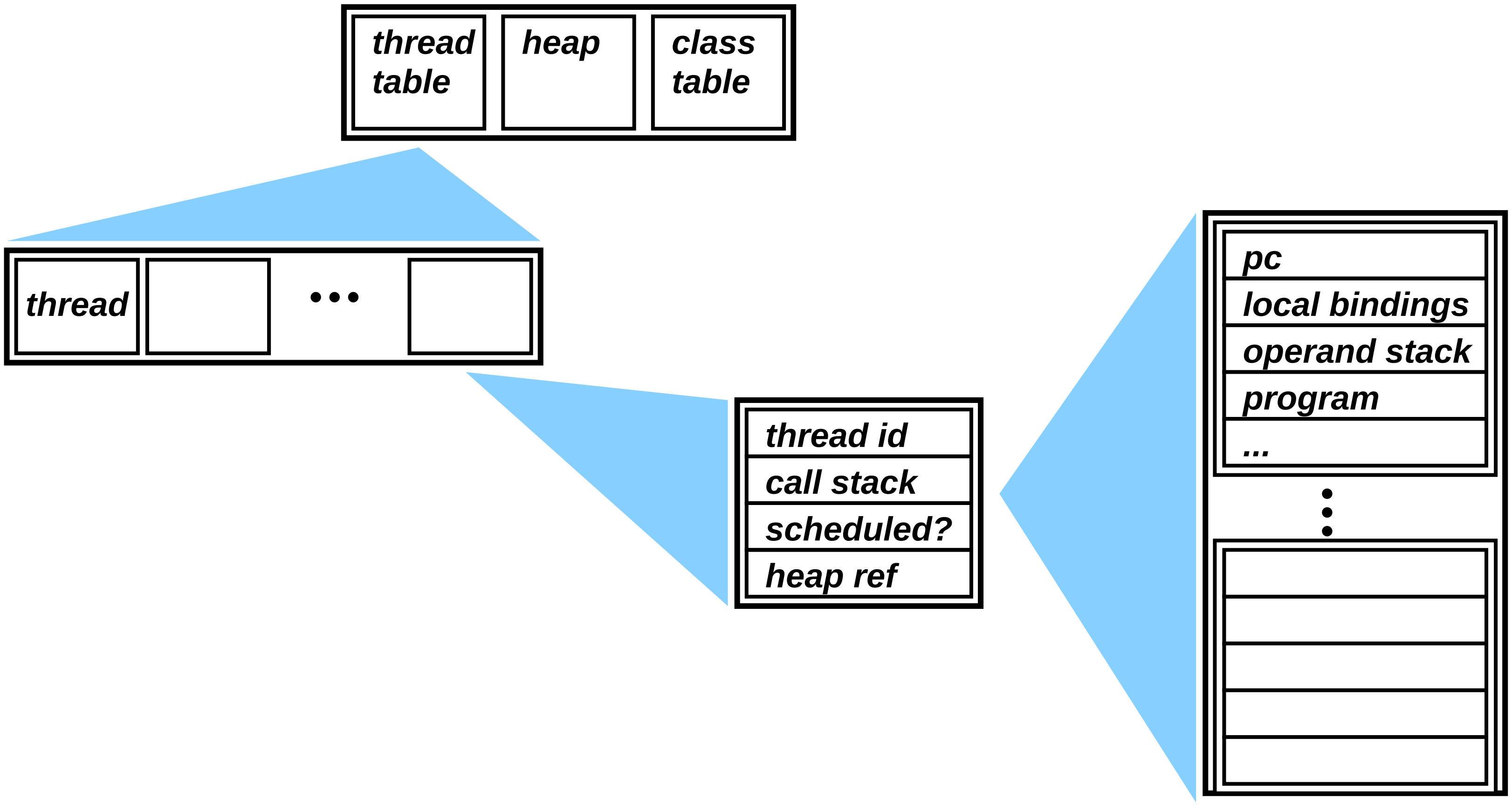
Our State: $< tt, hp, ct >$



```
(defun step (th s)
  (if (equal (call-stack-status th s)
             'SCHEDULED)
      (do-inst (next-inst th s) th s)
      s))
```

In our case, `th` is a thread identifier and is treated as a "signal."

Our State: $< tt, hp, ct >$



```
(defun do-inst (inst th s)
 (case (op-code inst)
  (AALOAD  (execute-AALOAD inst th s))
  (AASTORE (execute-AASTORE inst th s))
  (ALOAD   (execute-ALOAD inst th s))
  (ALOAD_0 (execute-ALOAD_X inst th s 0))
  (ALOAD_1 (execute-ALOAD_X inst th s 1))
  (ALOAD_2 (execute-ALOAD_X inst th s 2))
  (ALOAD_3 (execute-ALOAD_X inst th s 3))
  ...))
```

The JVM Spec from Sun

iload_0

Operation

Load `int` from local variable 0

Format

iload_0

Form

26 (0x1a)

Operand Stack

… $\Rightarrow$ …, value

Description

The local variable at 0 must contain an `int`.

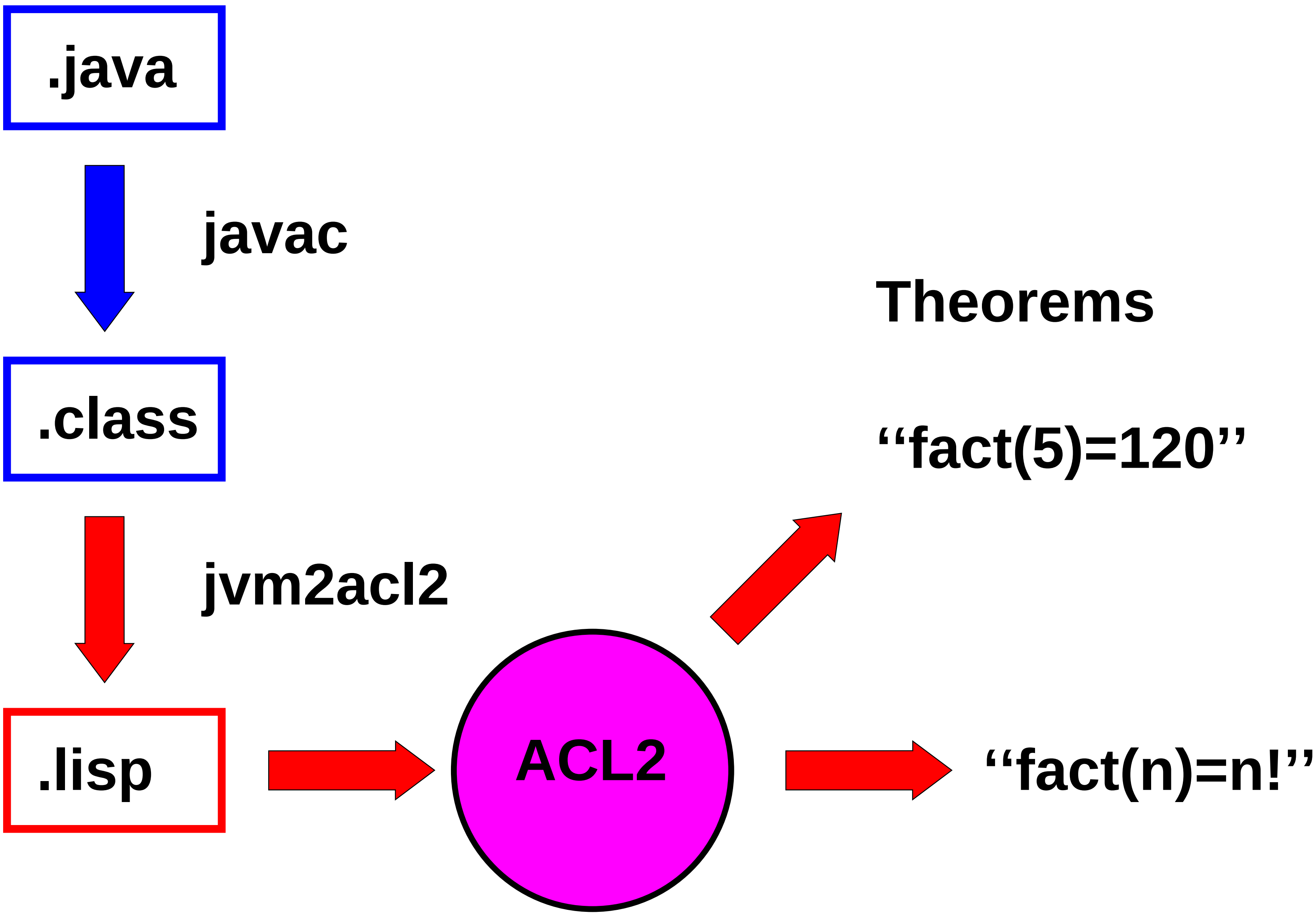
The value of the local variable at 0 is pushed onto the operand stack.

Note: `ILOAD_0`, ... `ILOAD_3` are 1-byte specializations of the 3-byte `ILOAD` n.

```
(defun execute-ILOAD (inst th s)
                                  ; inst = (ILOAD n)
 (let ((n (arg1 inst)))
  (modfy th s
  :pc (+ (inst-length inst)
         (pc (top-frame th s)))
  :stack (push (nth n
                    (locals (top-frame th s)))
               (stack (top-frame th s))))))
```

Some Java

```
class Demo {

  public static int fact(int n){
    if (n>0) {return n*fact(n-1);}
    else return 1;
  }
  public static void main(String[] args){
    int n = Integer.parseInt(args[0], 10);
    System.out.println(fact(n));
    return;
  } }
```



.java
javac
.class
jvm2acl2
.lisp
ACL2
Theorems
“fact(5)=120”
“fact(n)=n!”

Demo 3

Performance

The ACL2 function `run` is an *executable formal model* of the JVM.

On a 728 MHz processor, we get about 75K bytecodes/second.

With optimization, we get about 3M bytecodes/sec.

```
% java Demo 20
-2102132736

ACL2 >(acl2-Demo 20)
-2102132736

ACL2 >(! 20)
2432902008176640000

ACL2 >(int-fix (! 20))
-2102132736
```

Conjecture

Our Java `fact` method computes the twos-complement integer representedby the ow-order 32 bits of the actual factorial.

Demo 4

Pedagogy

What are the lessons here for undergraduate Computer Science majors?

- Mechanized formal analysis of digital artifacts is sometimes *possible* and *effective*.

- Formal specifications can serve as *prototypes* and *simulators*.

- Formal specification of correctness often requires *definition* of new concepts.

- Formal proof is facilitated by *structured code development* and *compositional reasoning*.

- There is an illuminating duality between *induction* and *recursion*.

- *Code analysis techniques* are separable from *language semantic techniques*. E.g., *operational semantics* can be used directly to support *Floyd-Hoare style code proofs*.

- *Tool support* for formal methods is available across a wide spectrum of applications.

- Formal methods is *not a panacea* but just one of the tools available to the system designer.

Undergraduate "Formal Methods" Courses at UT Austin

- 313K *Logic, Sets and Functions* – introduction to mathematical logic (year 1).

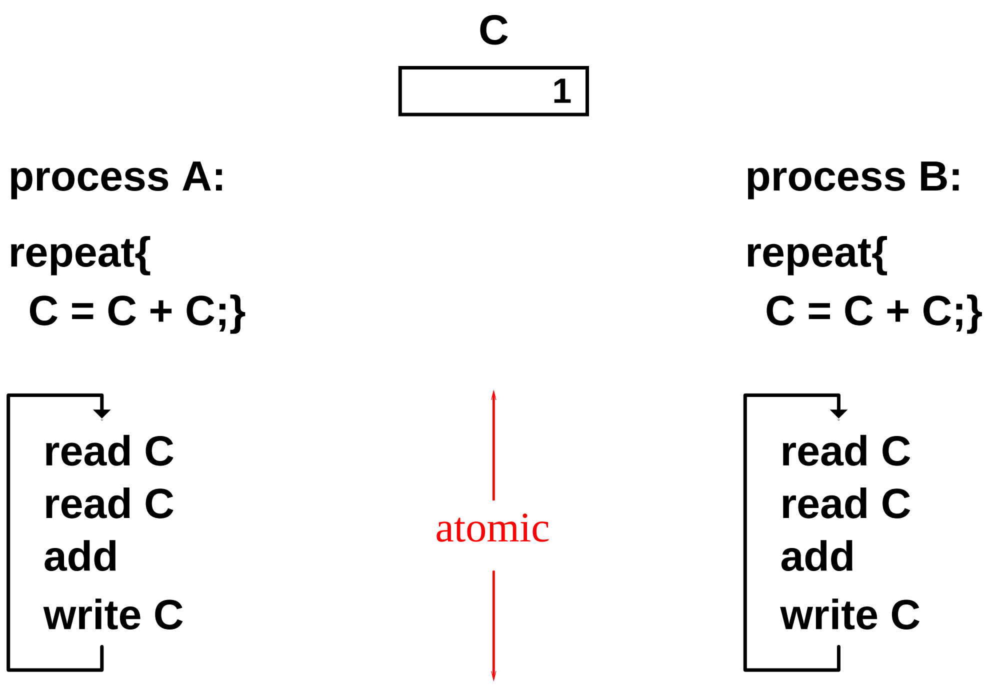
- 337 *Theory in Programming Practice* – illustrative examples of the use of formal analysis in program design (year 2).

- 336 *Analysis of Programs* – code analysis and proof techniques (year 2).

- 341 *Automata Theory* – automata and formal languages (year 3).
- 378 *Computer-Aided Verification* – model checking (year 4).
- 378 *Formal Model of the JVM* – theorem proving (year 4).

There are also a regular stream of topics classes on security, distributed programming, and hardware verification.

An Entertaining Puzzle: The Thread Game



Theorem? For every positive integer n there is an interleaving of A and B steps that produces C $= n$.

References

Computer-Aided Reasoning: An Approach, Kaufmann, Manolios, Moore, Kluwer Academic Publishers, 2000.

Computer-Aided Reasoning: ACL2 Case Studies, Kaufmann, Manolios, Moore (eds.), Kluwer Academic Publishers, 2000.

http://www.cs.utexas.edu/users/moore/acl2