Mechanized Operational Semantics

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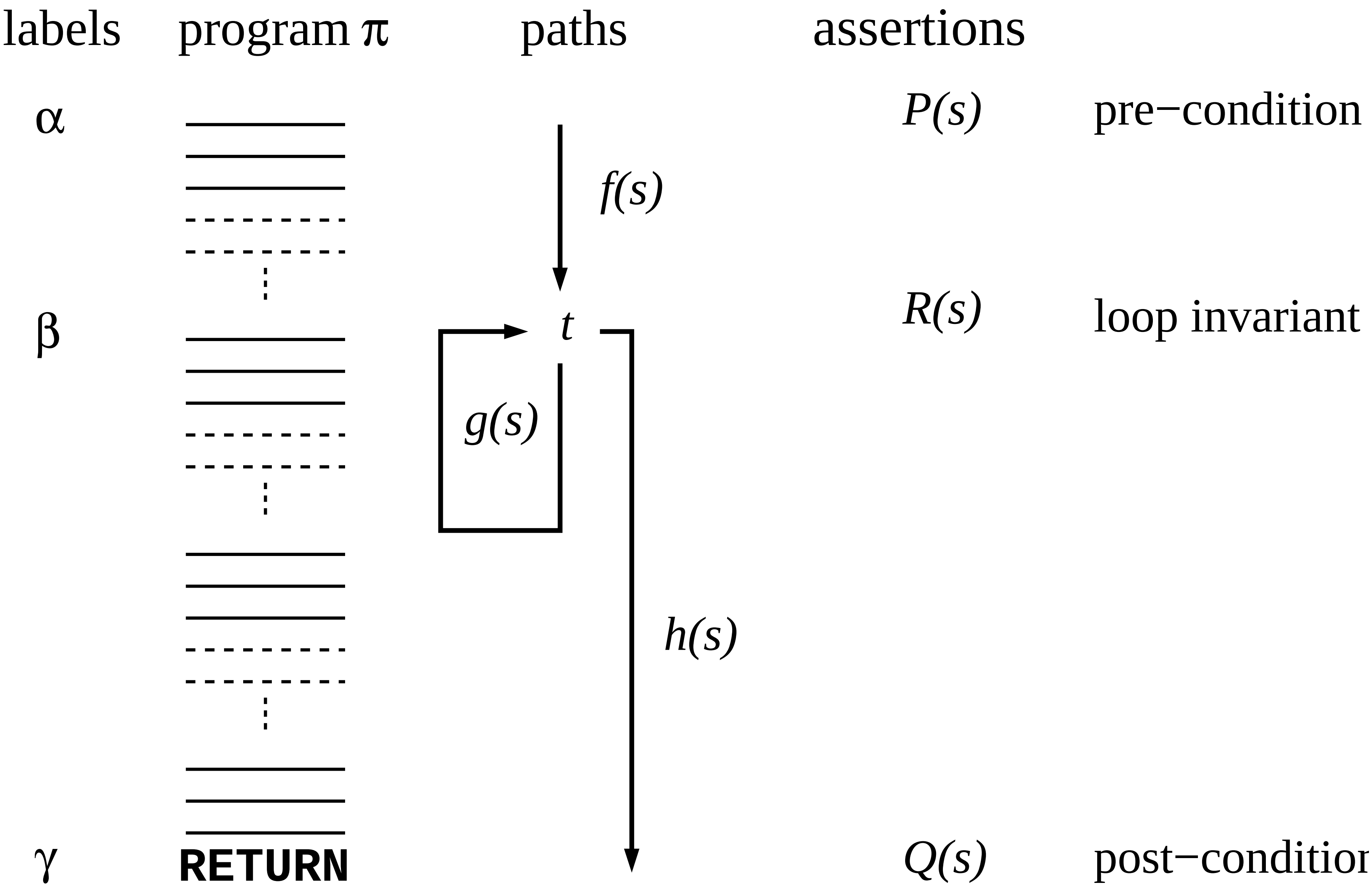
(Lecture 4: Inductive Invariant Proofs)



labels program π paths assertions

α

f(s)

P(s) pre−condition

β

t

g(s)

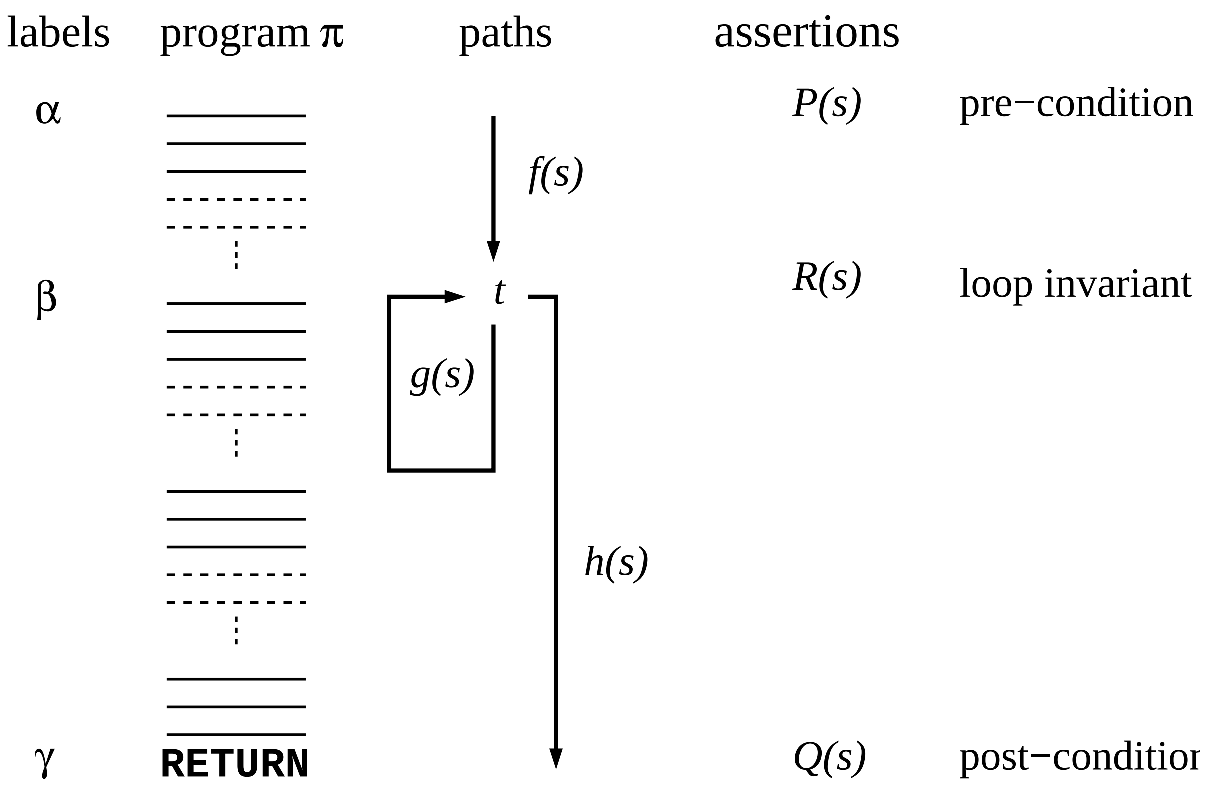
R(s) loop invariant

h(s)

γ **RETURN**

Q(s) post−condition

Conventional Mechanized Code Proofs



labels program π paths assertions

α $P(s)$ pre−condition

$f(s)$

β t $R(s)$ loop invariant

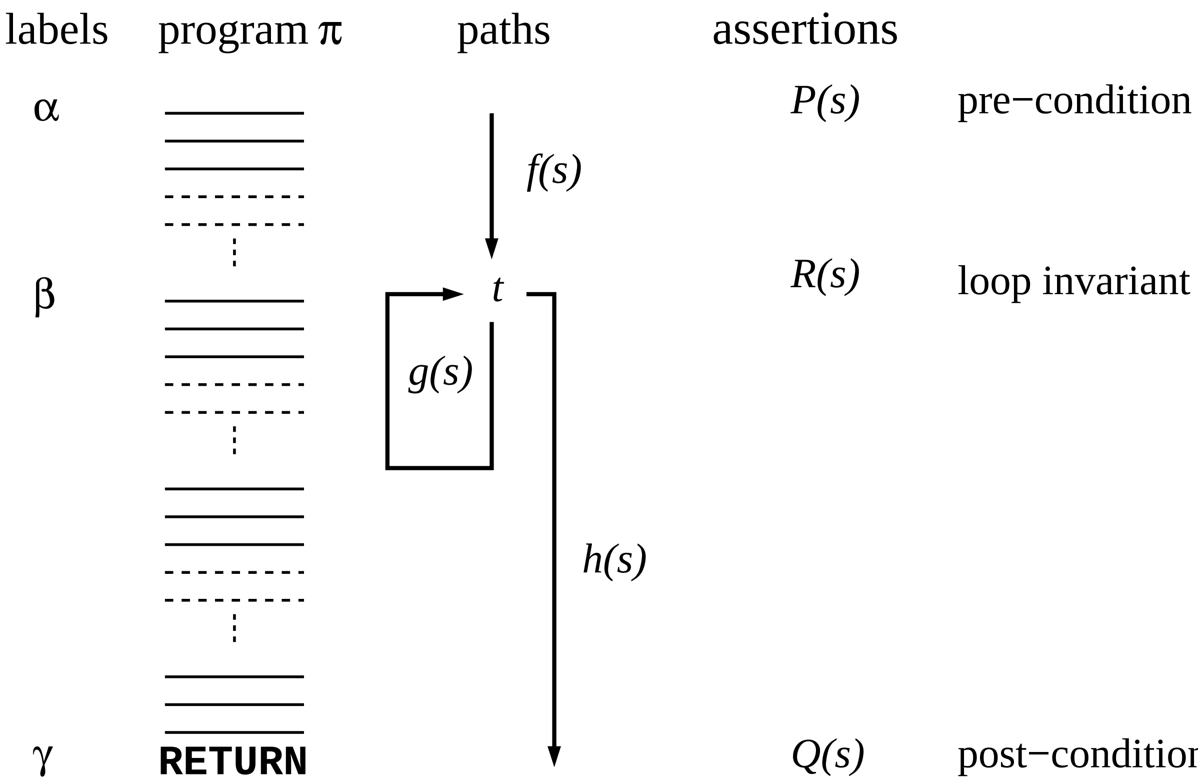
$g(s)$

$h(s)$

γ **RETURN** $Q(s)$ post−conditior

VC1. $P(s) \to R(f(s))$,

Conventional Mechanized Code Proofs



labels program π paths assertions

α *P(s)* pre−condition

f(s)

β *t* *R(s)* loop invariant

g(s)

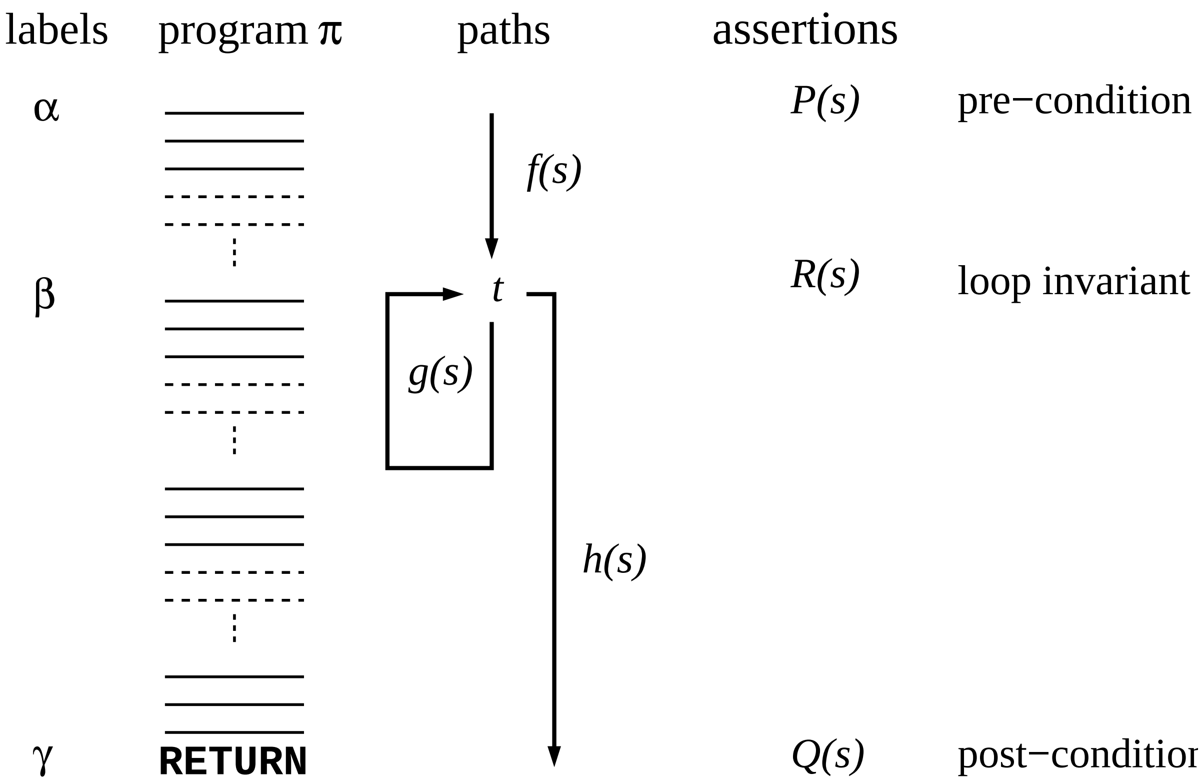
h(s)

γ **RETURN** *Q(s)* post−conditior

VC1. $P(s) \rightarrow R(f(s))$,

VC2. $R(s) \wedge t \rightarrow R(g(s))$, and

Conventional Mechanized Code Proofs



VC1. $P\,(s) \rightarrow R\,(f\,(s))$,

VC2. $R\,(s) \wedge t \rightarrow R\,(g\,(s))$, and

VC3. $R\,(s) \wedge \neg t \rightarrow Q\,(h(s))$.

Conventional Mechanized Code Proofs

The process by which proof obligations (*verification conditions* or *VCs*) are generated from the code is called *verification condition generation* and is performed by a *VCG* program.

Typically, VCGs simplify the VC “on-the-fly.”

Typically, the language semantics is coded into the VCG.

These are a common sources of errors.

Conventional Mechanized Code Proofs

To do conventional mechanized code proofs you need:

- a Hoare semantics
- a VCG (driven off the semantics)
- a theorem prover

Conventional Mechanized Code Proofs

To do conventional mechanized code proofs you need:

- an operational semantics
-
- a theorem prover

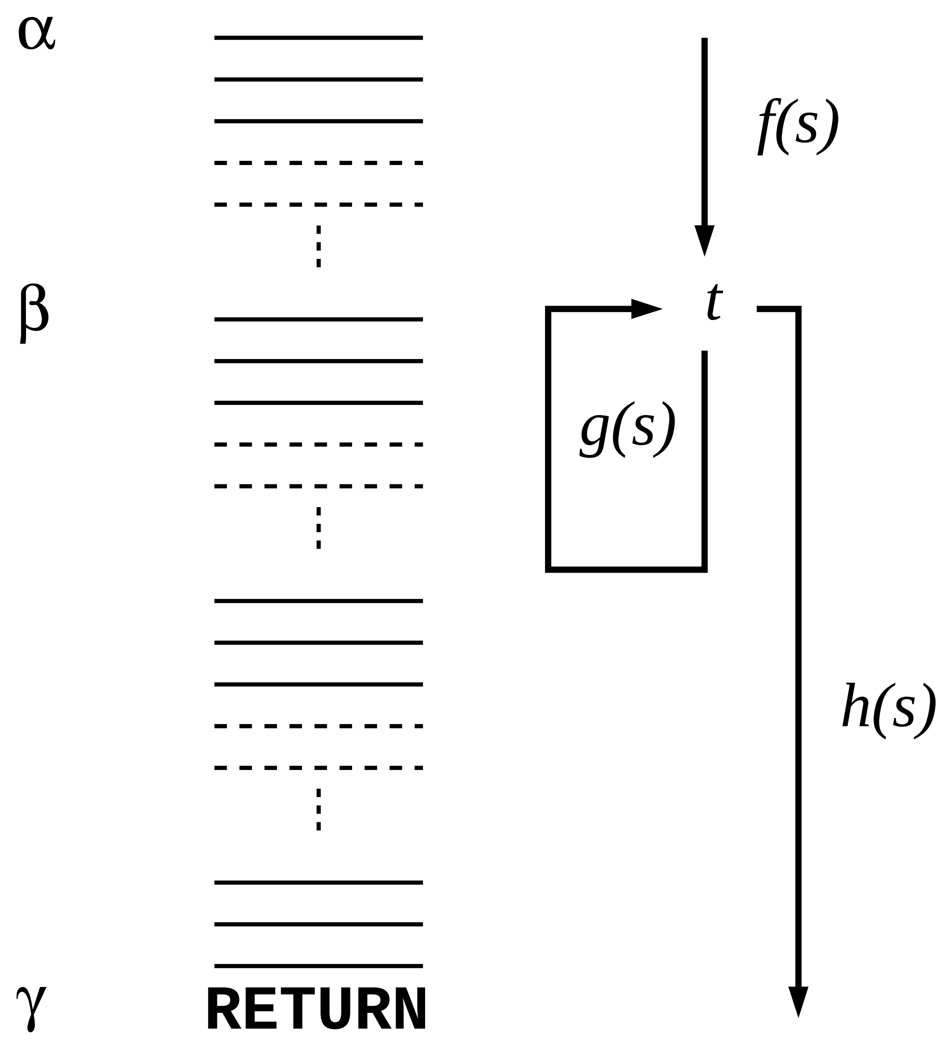
Operational Semantics (Revisited)

The *semantics* of the programming language may be given by a function run which “interprets” a program against some state and determines the “final” state.

$$run\,(k, s) = \begin{cases} s & \text{if } k = 0 \\ run\,(k - 1, step\,(s)) & \text{otherwise} \end{cases}$$

Here, $step$ is the single step state transition function.

labels program π paths assertions



$P(s)$ pre−condition

$R(s)$ loop invariant

$Q(s)$ post−conditio

We assume the program in s, π, does not change during execution.

Let s_0 be the initial state of program π.

$pc\left(s_0\right) = \alpha$

Let s_k denote $run\left(k, s_0\right)$.

Formally Stated Correctness Theorems

Total:

$$\exists k : P\,(s_0) \rightarrow (Q(s_k) \wedge pc(s_k) = \gamma).$$

$$\exists k : P\,(s_0) \rightarrow (Q(run\,(k, s_0)) \wedge \ldots .$$

This is sometimes stated without the quantifier as

$$P\,(s_0) \rightarrow (Q(run\,(sched\,(s_0), s_0)) \wedge \ldots).$$

Partial:

$$P\,(s_0)\ \wedge\ pc(s_k) = \gamma \rightarrow Q(s_k).$$

Disadvantage of Direct Proofs

Direct proofs of program properties can be complicated (or at least appear so) because of the presence of the interpreter, the program counter, the entire machine state, and the need to define a schedule function.

The inductive assertion method produces such nice proof obligations!

Conundrum

Can you prove

$$P\,(s_0)\ \wedge\ pc(s_k) = \gamma \rightarrow Q(s_k).$$

directly – where the only heavy-duty proof work is proving the verification conditions?

Do you need a trusted VCG?

Can you make the automatic proof attempt *generate* the standard verification conditions from the operational semantics?

Caveat

The observations I make below are not deep, but I think they have important practical implications:

Theorem: $P(s_0) \wedge pc(s_k) = \gamma \rightarrow Q(s_k)$

Proof: Define

$$Inv(s) \equiv \begin{cases} P(s) & \text{if } pc(s) = \alpha \\ R(s) & \text{if } pc(s) = \beta \\ Q(s) & \text{if } pc(s) = \gamma \\ Inv(step(s)) & \text{otherwise} \end{cases}$$

(Actually, we assert "$prog(s) = \pi$" at α, β and γ, but we omit that here by our convention that the program is always π.)

Theorem: $P(s_0) \wedge pc(s_k) = \gamma \rightarrow Q(s_k)$

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Objection: Is this definition consistent? Yes: Every tail-recursive definition is witnessed by a total function. (Manolios and Moore, 2000)

Theorem: $P(s_0) \wedge pc(s_k) = \gamma \rightarrow Q(s_k)$

Proof: Define

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Assume we've proved

$$Inv(s) \rightarrow Inv(step(s)).$$

(We'll see the proof in a moment.)

Theorem: $P(s_0) \wedge pc(s_k) = \gamma \rightarrow Q(s_k)$

Proof: Define

$$Inv(s) \equiv \begin{cases} P(s) & \text{if } pc(s) = \alpha \\ R(s) & \text{if } pc(s) = \beta \\ Q(s) & \text{if } pc(s) = \gamma \\ Inv(step(s)) & \text{otherwise} \end{cases}$$

$Inv(s_0) \rightarrow Inv(s_k)$ *(By induction)*

Theorem: $P\left(s_0\right) \wedge pc\left(s_k\right) = \gamma \rightarrow Q\left(s_k\right)$

Proof: Define

$$Inv\left(s\right) \equiv \begin{cases} P\left(s\right) & \text{if } pc\left(s\right) = \alpha \\ R\left(s\right) & \text{if } pc\left(s\right) = \beta \\ Q\left(s\right) & \text{if } pc\left(s\right) = \gamma \\ Inv\left(step\left(s\right)\right) & \text{otherwise} \end{cases}$$

$Inv\left(s_0\right) \rightarrow Inv\left(s_k\right)$

$pc\left(s_0\right) = \alpha \qquad \textit{(By construction)}$

Theorem: $P\left(s_0\right) \wedge pc\left(s_k\right) = \gamma \rightarrow Q\left(s_k\right)$

Proof: Define

$$Inv\left(s\right) \equiv \begin{cases} P\left(s\right) & \text{if } pc\left(s\right) = \alpha \\ R\left(s\right) & \text{if } pc\left(s\right) = \beta \\ Q\left(s\right) & \text{if } pc\left(s\right) = \gamma \\ Inv\left(step\left(s\right)\right) & \text{otherwise} \end{cases}$$

$P\left(s_0\right) \rightarrow Inv\left(s_k\right)$

Theorem: $P(s_0) \wedge pc(s_k) = \gamma \rightarrow Q(s_k)$

Proof: Define

$$Inv(s) \equiv \begin{cases} P(s) & \text{if } pc(s) = \alpha \\ R(s) & \text{if } pc(s) = \beta \\ Q(s) & \text{if } pc(s) = \gamma \\ Inv(step(s)) & \text{otherwise} \end{cases}$$

$P(s_0) \rightarrow Inv(s_k)$

$P(s_0)$ *(Given)*

Theorem: $P(s_0) \wedge pc(s_k) = \gamma \rightarrow Q(s_k)$

Proof: Define

$$Inv(s) \equiv \begin{cases} P(s) & \text{if } pc(s) = \alpha \\ R(s) & \text{if } pc(s) = \beta \\ Q(s) & \text{if } pc(s) = \gamma \\ Inv(step(s)) & \text{otherwise} \end{cases}$$

$Inv(s_k)$

Theorem: $P(s_0) \wedge pc(s_k) = \gamma \rightarrow Q(s_k)$

Proof: Define

$$Inv(s) \equiv \begin{cases} P(s) & \text{if } pc(s) = \alpha \\ R(s) & \text{if } pc(s) = \beta \\ Q(s) & \text{if } pc(s) = \gamma \\ Inv(step(s)) & \text{otherwise} \end{cases}$$

$Inv(s_k)$

$pc(s_k) = \gamma$ $\quad$ *(Given)*

Theorem: $P(s_0) \wedge pc(s_k) = \gamma \rightarrow Q(s_k)$

Proof: Define

$$Inv(s) \equiv \begin{cases} P(s) & \text{if } pc(s) = \alpha \\ R(s) & \text{if } pc(s) = \beta \\ Q(s) & \text{if } pc(s) = \gamma \\ Inv(step(s)) & \text{otherwise} \end{cases}$$

$Q(s_k)$

Q.E.D.

So it's trivial to prove the theorem

$$P(s_0) \wedge pc(s_k) = \gamma \rightarrow Q(s_k)$$

if we can prove

$$Inv(s) \rightarrow Inv(step(s)).$$

$$Inv\left(s\right) \equiv \begin{cases} P\left(s\right) & \text{if } pc\left(s\right) = \alpha \\ R\left(s\right) & \text{if } pc\left(s\right) = \beta \\ Q\left(s\right) & \text{if } pc\left(s\right) = \gamma \\ Inv\left(step\left(s\right)\right) & \text{otherwise} \end{cases}$$

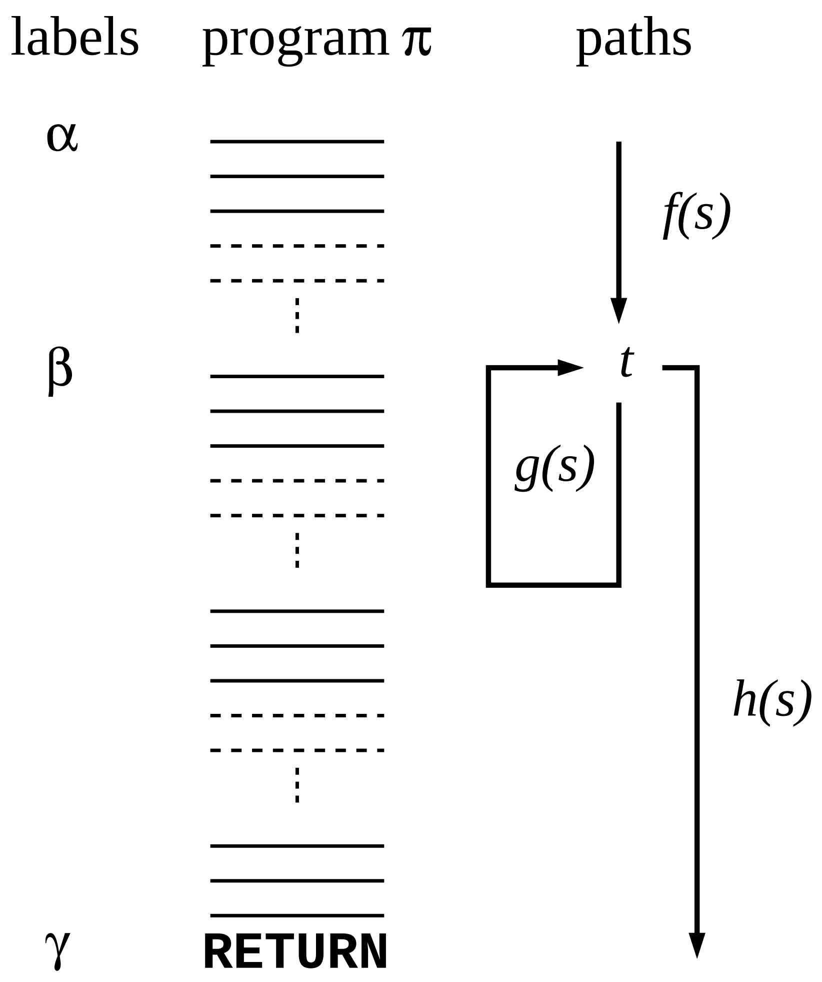
$Inv\,(s) \to Inv\,(step\,(s))$

Proof.

Expanding $Inv\,(s)$ generates four cases:

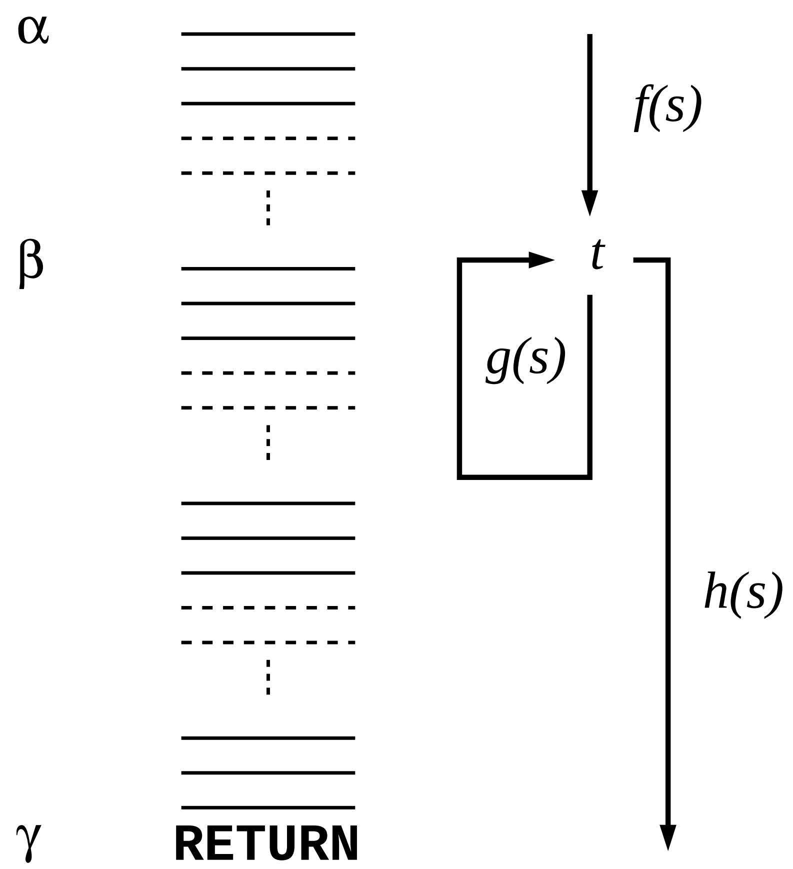
Case $pc\,(s) = \alpha$:
Case $pc\,(s) = \beta$:
Case $pc\,(s) = \gamma$:
Case *otherwise*:

$$Inv\,(s) \rightarrow Inv\,(step\,(s)) \qquad [\text{Case } pc(s) = \alpha]$$



assertions	
P(s)	pre−condition
R(s)	loop invariant
Q(s)	post−condition

$P(s) \to Inv(step(s))$ [Case $pc(s) = \alpha$]



labels program π paths

α

β

γ RETURN

f(s)

t

g(s)

h(s)

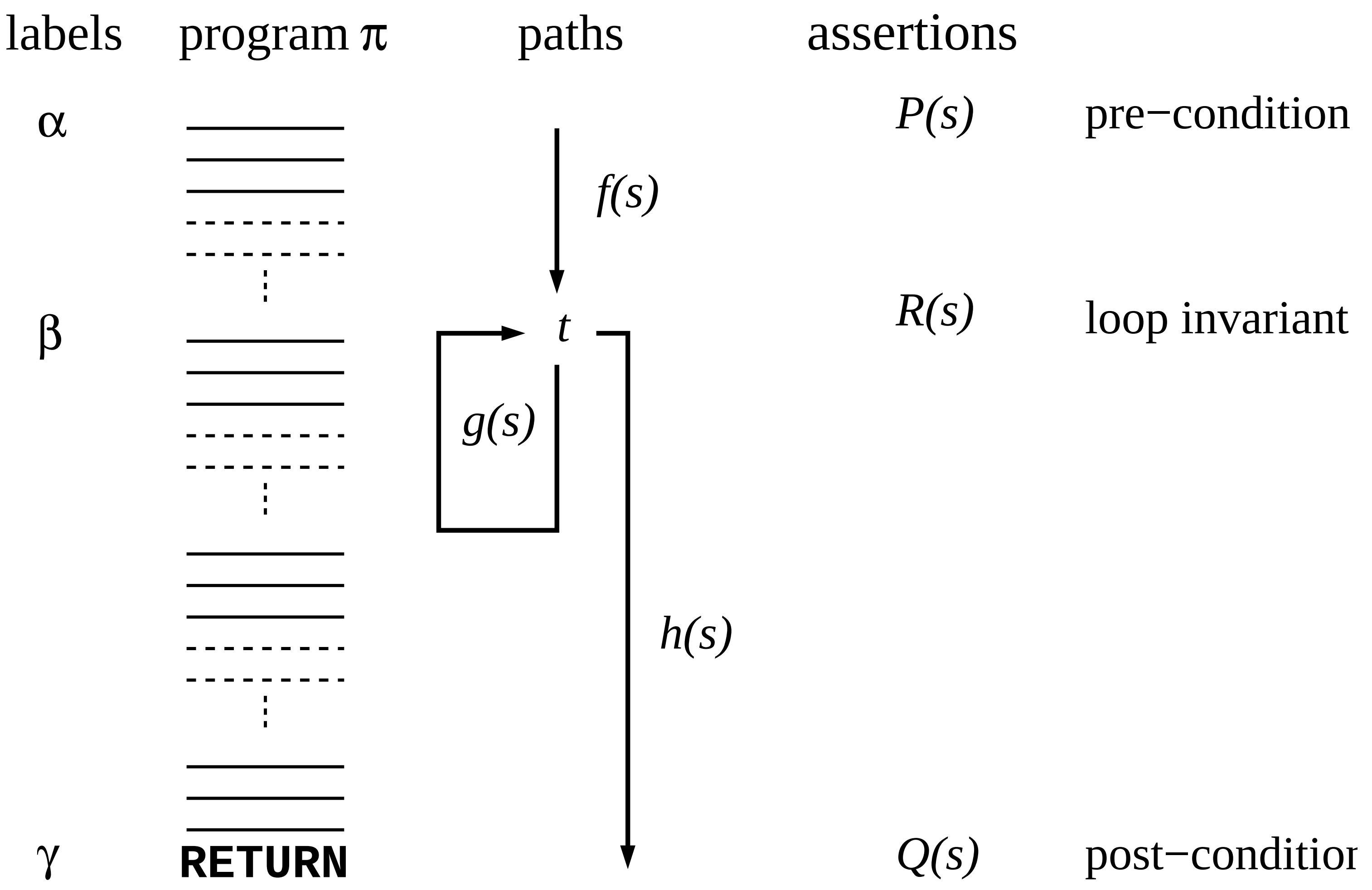
assertions	
P(s)	pre−condition
R(s)	loop invariant
Q(s)	post−conditior

$$Inv\,(s) \equiv \begin{cases} P\,(s) & \text{if } pc\,(s) = \alpha \\ R\,(s) & \text{if } pc\,(s) = \beta \\ Q\,(s) & \text{if } pc\,(s) = \gamma \\ Inv\,(step\,(s)) & \text{otherwise} \end{cases}$$

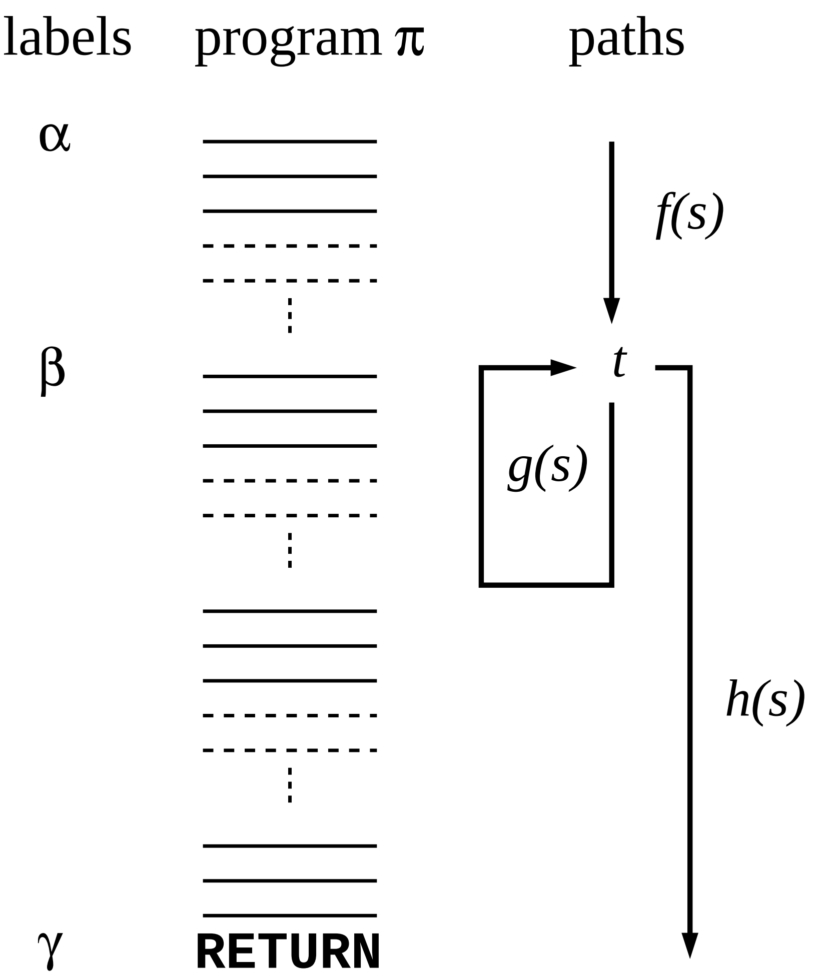
$$Inv\,(s) = Inv\,(step\,(s)) = Inv\,(step\,(step\,(s)))\ldots$$

as long as the $pc \notin \{\alpha, \beta, \gamma\}$.

$P(s) \rightarrow Inv(step(s))$ [Case $pc(s) = \alpha$]

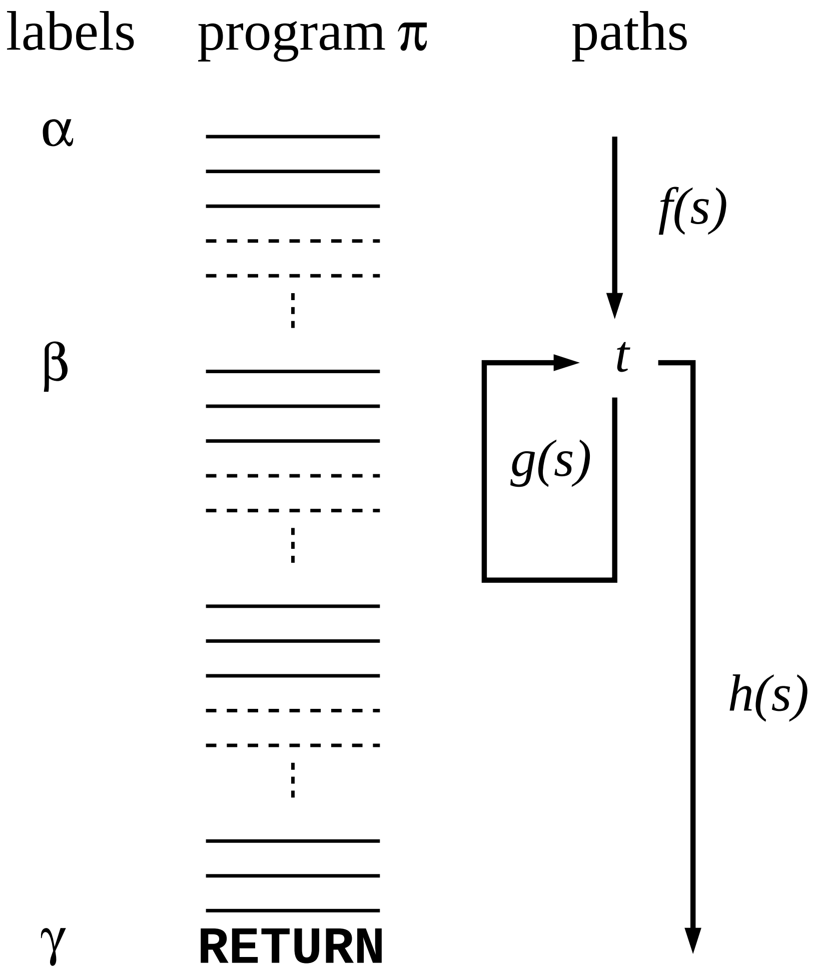


$P(s) \rightarrow R(f(s))$ [Case $pc(s) = \alpha$]



assertions	
$P(s)$	pre−condition
$R(s)$	loop invariant
$Q(s)$	post−conditio

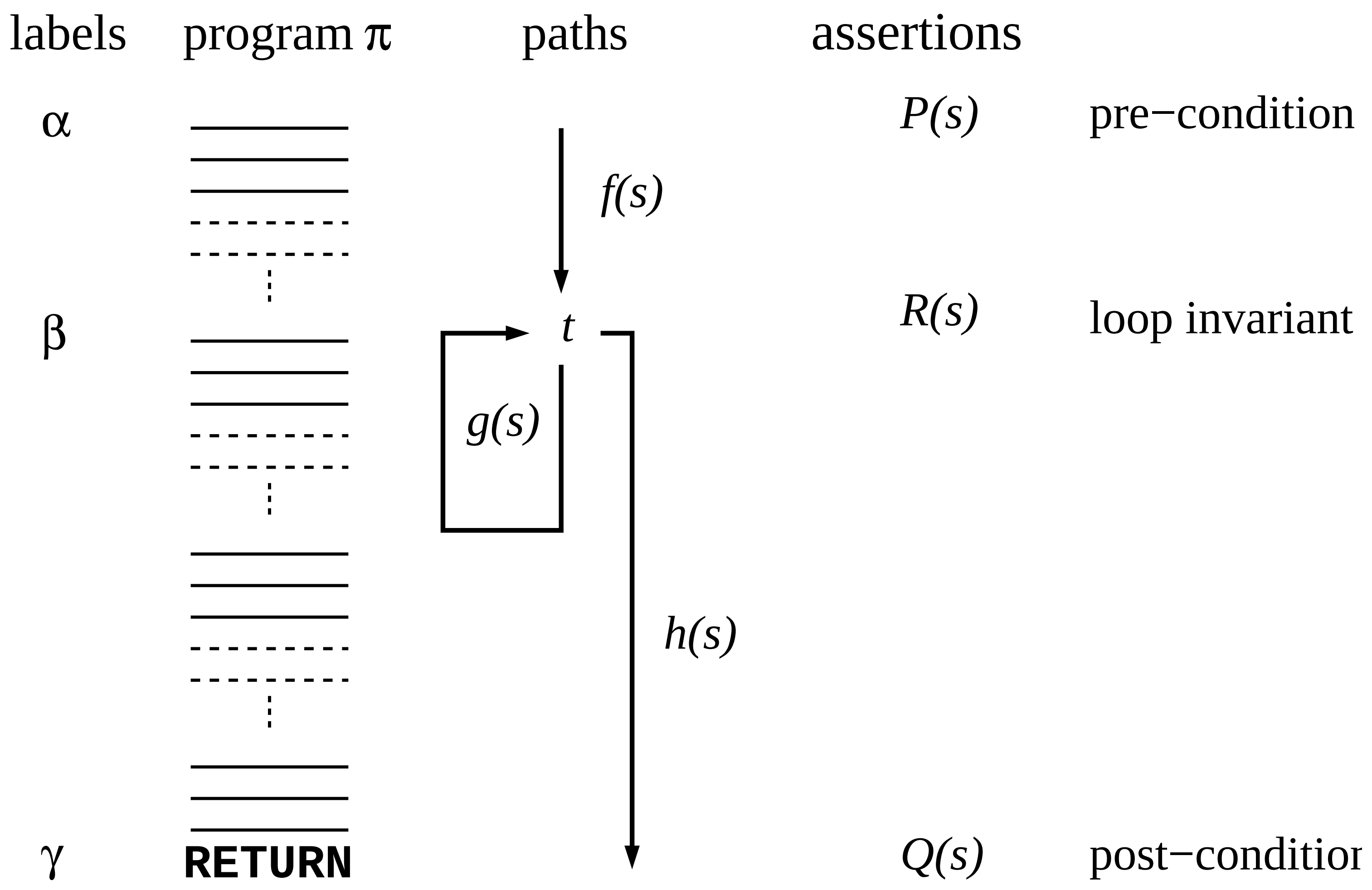
$Inv\,(s) \rightarrow Inv\,(step\,(s))$ [Case $pc(s) = \beta$]



assertions	
$P(s)$	pre−condition
$R(s)$	loop invariant
$Q(s)$	post−condition

$$(R(s) \wedge t \rightarrow R(g(s)))$$
$$(R(s) \wedge \neg t \rightarrow Q(h(s)))$$

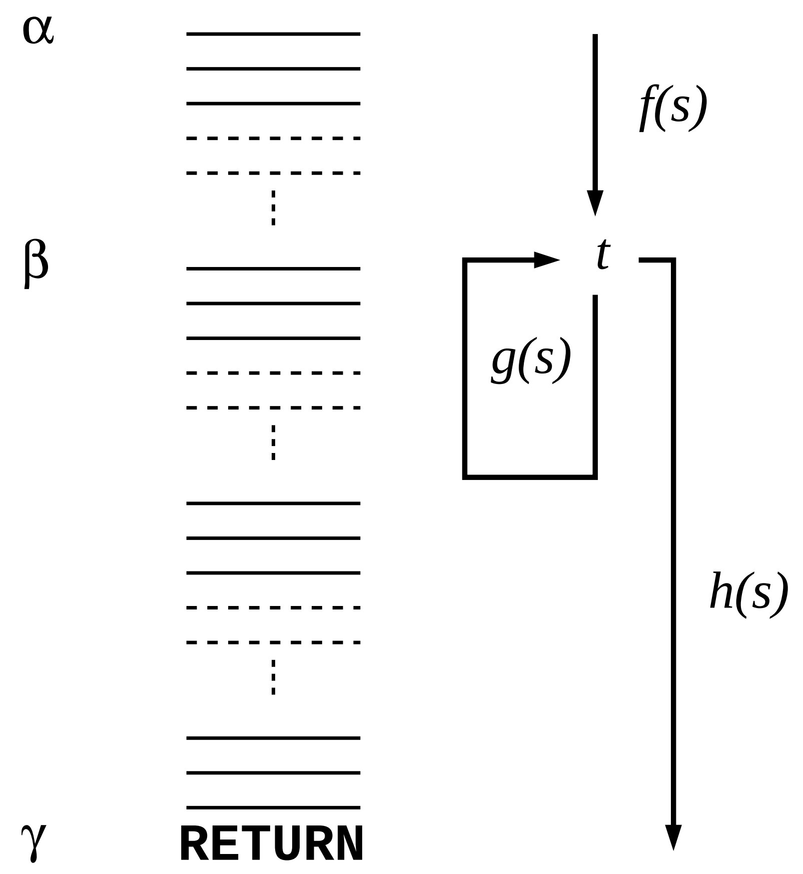
[Case $pc(s) = \beta$]



P(s)	pre−condition
R(s)	loop invariant
Q(s)	post−conditior

$Inv(s) \rightarrow Inv(step(s))$ [Case $pc(s) = \gamma$]

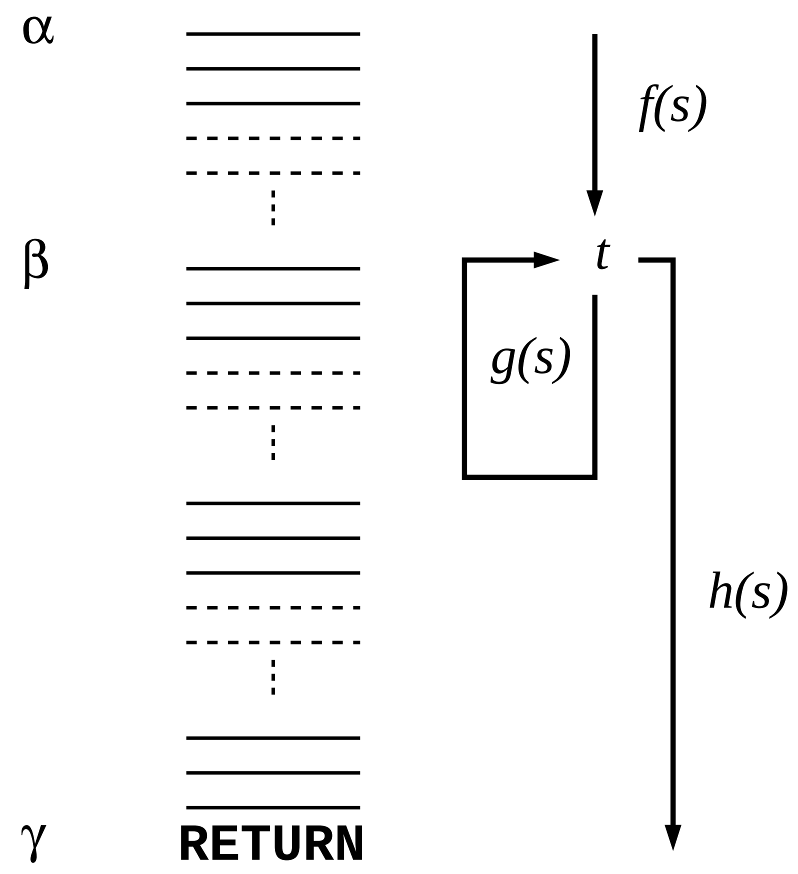
labels program π paths assertions



$P(s)$	pre−condition
$R(s)$	loop invariant
$Q(s)$	post−condition

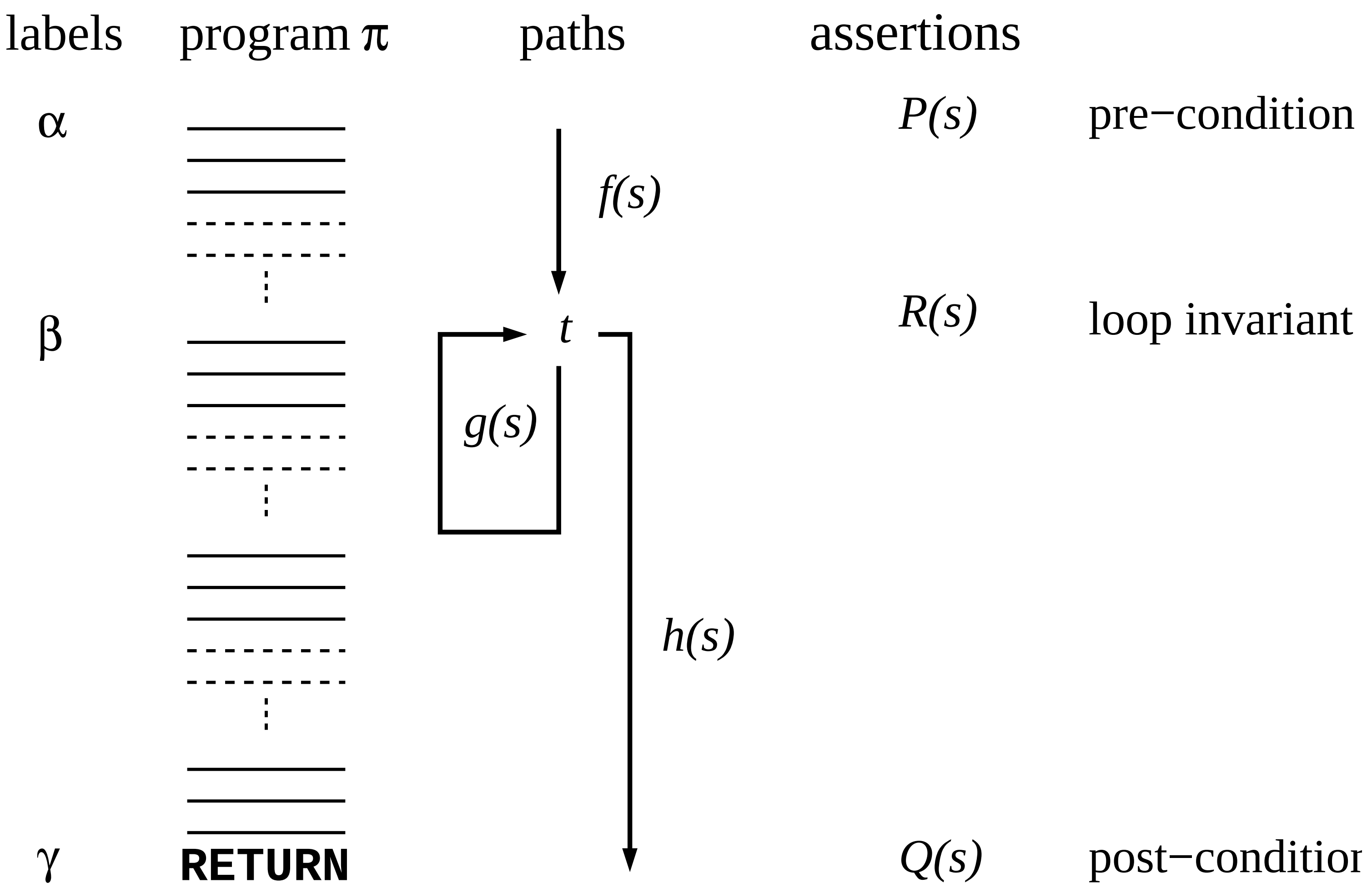
$Inv(s) \rightarrow Inv(s)$ [Case $pc(s) = \gamma$]

labels program π paths assertions

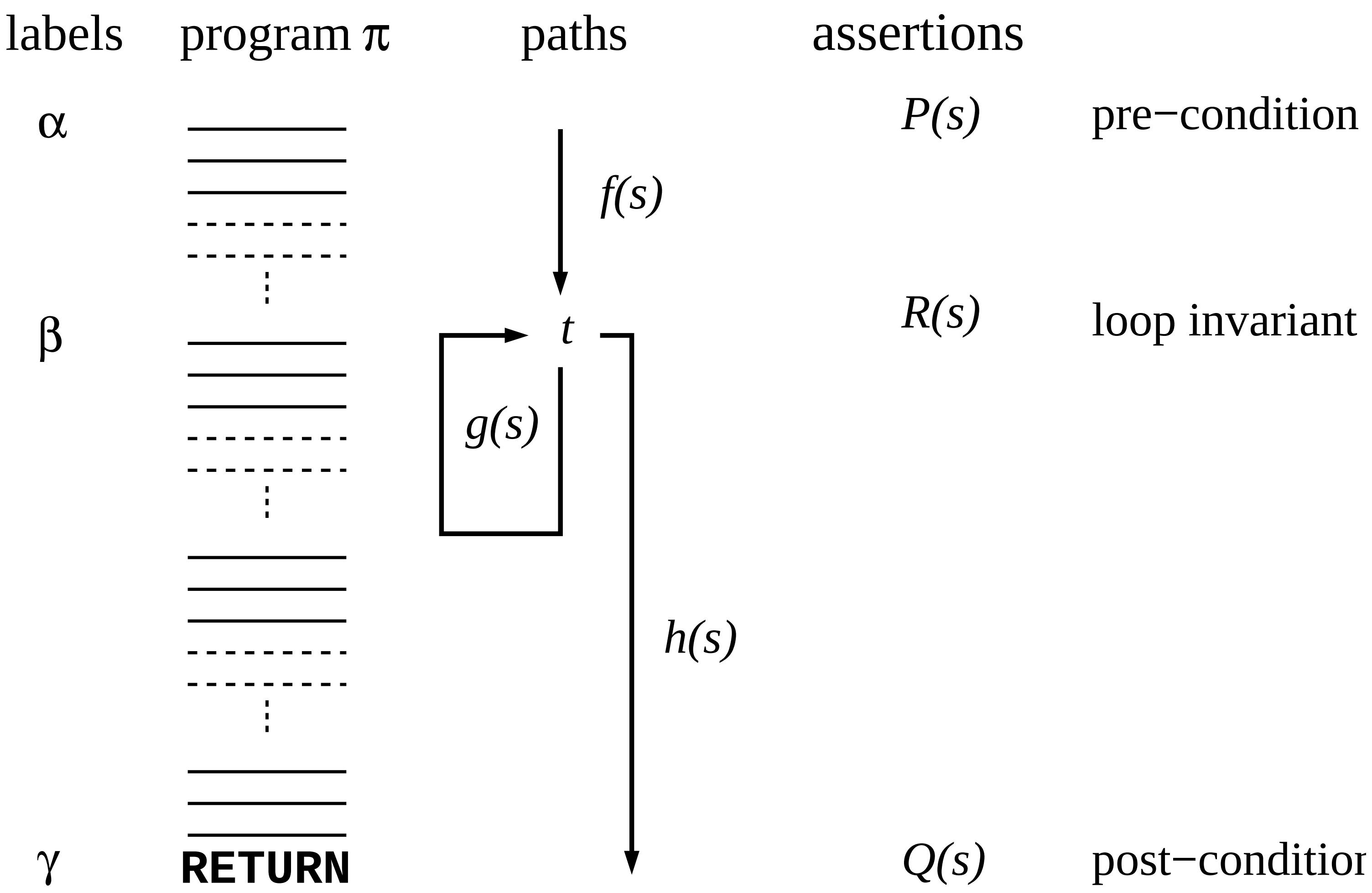


P(s)	pre−condition
R(s)	loop invariant
Q(s)	post−conditior

$Inv\,(s) \rightarrow Inv\,(step\,(s))$ [Case *otherwise*]



$Inv\,(step\,(s)) \rightarrow Inv\,(step\,(s))$ [Case *otherwise*]



labels	program π	paths	assertions	
α		$f(s)$	$P(s)$	pre−condition
β		t $g(s)$ $h(s)$	$R(s)$	loop invariant
γ	**RETURN**		$Q(s)$	post−conditio

Recap: Given the definition of Inv, the "natural" proof of

$Inv\,(s) \rightarrow Inv\,(step\,(s))$

generates the standard verification conditions

VC1. $P\,(s) \rightarrow R\,(f\,(s))$,
VC2. $R\,(s) \wedge t \rightarrow R\,(g\,(s))$, and
VC3. $R\,(s) \wedge \neg t \rightarrow Q\,(h(s))$
as subgoals from the operational semantics!

It generates no other non-trivial proof obligations.

The VCs are simplified as they are generated.

Demo 1

Discussion

We did not write a VCG for M1.

The VCs were generated directly from the operational semantics by the theorem prover.

Since VCs are generated by proof, the paths explored and the VCs generated are sensitive to the pre-condition specified.

The VCs are simplified (and possibly proved) by the same process.

We did not count instructions or define a schedule.

We did not constrain the inputs so that the program terminated.

Indeed, we can deal with non-terminating programs.

Demo 2

Total Correctness via Inductive Assertions

We have also handled total correctness via the VCG approach.

An ordinal measure is provided at each cut point and the VCs establish that it decreases upon each arrival at the cut point.

Schedule functions can be automatically generated and admitted from such proofs.

Primary Citation

J S. Moore, “Inductive Assertions and Operational Semantics,” *CHARME 2003*, D. Geist (Ed.), Springer Verlag LNCS 2860, pp. 289–303, 2003.

Other Examples

Nested loops are handled exactly as by standard VCG methods.

```
public static int tfact(int n){  /* Factorial by repeated addition.     */
    int i = 1;                   /* Verified using inductive assertions */
    int b = 1;                   /* by Alan Turing, 1949.               */
    while (i<=n){
        int j = 1;
        int a = b;
        while (j < i) {
            b = a+b;
            j++;
        };
        i++;
    };
    return b;
}
```

Recursive methods can be handled.

```
public static int fact(int n){
  if (n>0)
    {return n*fact(n-1);}
  else return 1;
}
```

To handle recursive methods we

- modify run to terminate upon top-level `return`, and
- add a standard invariant about the shape of the call stack.

Conclusion

If you have

- a theorem prover and
- a formal operational semantics,

you can prove formally stated *partial program correctness* theorems using *inductive assertions* without building or verifying a VCG.

Related Work

P. Y. Gloess, "Imperative Program Verification in PVS," École Nationale Supérieure Électronique, Informatique et Radiocommunications de Bordeaux, 1999.

P. Homeier and D. Martin, "A Mechanically Verified Verification Condition Generator," *The Computer Journal*, **38**(2), pp. 131–141, July 1995.

P. Manolios and J Moore, "Partial Functions in ACL2," *JAR* 2003.

J. Matthews, J S. Moore, S. Ray, and D. Vroon: "Verification Condition Generation via Theorem Proving," to appear in M. Hermann and A. Voronkov, editors, *Proceedings of the 13th International Conference on Logic for Programming, Artifical Intelligence, and Reasoning* (LPAR 2006), Phnom Penh, Cambodia, November 2006, Springer-Verlag.

Next Time

a much more interesting correctness proof